

## Syllabus

Course Number and Title: Math 1320-001. Engineering Calculus II.

Textbook: Located on Canvas "Bookshelf" tab.

Semester and Year: Summer 2021

Zoom Link For Course: Linked on Canvas

Instructor: Kristen Lee

Email: [klee@math.utah.edu](mailto:klee@math.utah.edu) and Canvas Mail.

Accessibility and Support: Canvas Mail is my preferred form of communication. I will answer emails Monday-Friday and not on Saturday or Sunday. Please allow 24 hours for me to respond to any emails you send.

Office Hours: Mondays, 1:00 pm-2:00 pm Mountain Standard Time

Wednesdays: 9:45 am-10:45 am Mountain Standard Time

## Course Structure

- Students will receive lectures through zoom on Monday, Tuesday, Wednesday, and Friday of each week aside from holidays.
- Students will participate in lab on Zoom on Thursday of each week.
- There is a quiz every Friday except on exam days.
- All homework associated with lectures in a given week will be due the following Wednesday on Canvas at 11:59 pm Mountain Standard Time.
- The lab from a given week will be due the Wednesday of the following week at 11:59 pm Mountain Standard Time.

## Exam Dates

- Midterm 1: Friday, June 11 during class time on zoom.
- Midterm 2: Friday, July 2 during class time on zoom.
- Midterm 3: Thursday, July 22 during class time on zoom.
- Final Exam: Thursday, August 5, 7:30-9:30.

## Other important Dates

- Memorial Day: May 31, No class
- Independence Day (observed): July 5, No Class
- Pioneer Day (observed): July 23, No Class
- Last day drop or elect credit/no credit: Wednesday, May 26
- Last day to withdrawal from course: Friday, June 25
- Last day to reverse credit/no credit option: Friday, July 30
- Last day of classes: Wednesday, August 4

### Grade Breakdown

- Announcement Quizzes-----2%
- Homework-----8%
- Labs-----15%
- Quizzes-----10%
- Midterms-----45%
- Final-----20%

It is the student's responsibility to ensure the accuracy of all recorded homework, quizzes, online assignments, and exam grades. Also you should keep as record all your graded assignments. If you see any error in your grades on Canvas, reach out to the instructor as soon as possible, or at the latest within two weeks from when the assignment was returned.

### Grading Scale:

A [93-100),	B [83-87),	C [73-77),	D [60-66),
A- [90-93),	B- [80-83),	C- [70-73),	D- [55-60),
B+ [87-90),	C+ [77-80),	D+ [66-70),	E [0-55).

### Extra Credit

Extra credit is offered throughout the course. More info in Canvas.

### Late Work Policy

- I don't accept late work in this course
- The lowest 8 homework assignments will be dropped
- The lowest 2 quizzes will be dropped
- The lowest 2 labs will be dropped
- Students are not permitted to take exams late without a 12% penalty.
- If you miss an exam you will need to make it up in two school days.

### ACADEMIC CODE OF CONDUCT

Students are encouraged to review the Student Code for the University of Utah: <https://regulations.utah.edu/academics/6-400.php>. In order to ensure that the highest standards of academic conduct are promoted and supported at the University, students must adhere to generally accepted standards of academic honesty, including but not limited to refraining from cheating, plagiarizing, research misconduct, misrepresenting

one's work, and/or inappropriately collaborating. A student who engages in academic misconduct as defined in Part I.B. may be subject to academic sanctions including but not limited to a grade reduction, failing grade, probation, suspension or dismissal from the program or the University, or revocation of the student's degree or certificate. Sanctions may also include community service, a written reprimand, and/or a written statement of misconduct that can be put into an appropriate record maintained for purposes of the profession or discipline for which the student is preparing. If you have a specific policy, indicate it here (example: Incidents of academic misconduct (e.g. cheating, plagiarizing, misrepresenting one's work, and/or inappropriately collaborating on exams) will be subject to penalty per Section V of Policy 6-400, the Student Code. Incidents of academic dishonesty on homework assignments will result in a minimum penalty of a full letter-grade reduction and up to a failing grade (E) for the course. Incidents of academic dishonesty on exams will result in a minimum penalty of a failing grade (E) for the course, and the incident(s) will be referred to the dean of your major-department college for possible further sanction.).

#### ADDITIONAL POLICIES AND RESOURCES

**Inclusivity Statement:** It is my intent that students from all diverse backgrounds and perspectives be well served by this course, that students' learning needs be addressed both in and out of class, and that the diversity that students bring to this class be viewed as a resource, strength and benefit. It is my intent to present materials and activities that are respectful of diversity: age, color, disability, gender, gender identity, gender expression, national origin, political affiliation, race, religion, sexual orientation, and veteran status, and other unique identities. gender, sexuality, disability, age, socioeconomic status, ethnicity, race, culture, and other unique identities. Your suggestions are encouraged and appreciated. Please let me know ways to improve the effectiveness of the course for you personally or for other students or student groups. In addition, if any of our class meetings conflict with your religious events, please let me know so that we can make arrangements for you.

**Discrimination and Harassment:** If you or someone you know has been harassed or assaulted, you are encouraged to report it to the Title IX Coordinator in the Office of Equal Opportunity and Affirmative Action, 135 Park Building, 801-581-8365, or Office of the Dean of Students, 270 Union Building, 801-581-7066. To report to the police, contact the Department of Public Safety, 801-585-2677(COPS). Please see Student Bill of Rights, section E <http://regulations.utah.edu/academics/6-400.php>. I will listen and believe you if someone is threatening you.

**Names/Pronouns:** Canvas allows students to change the name that is displayed AND allows them to add their pronouns to their Canvas name. Class rosters are provided to

the instructor with the student's legal name as well as "Preferred first name" (if previously entered by you in the Student Profile section of your CIS account, which managed can be managed at any time). While CIS refers to this as merely a preference, I will honor you by referring to you with the name and pronoun that feels best for you in class or on assignments. Please advise me of any name or pronoun changes so I can help create a learning environment in which you, your name, and your pronoun are respected. If you need any assistance or support, please reach out to the LGBT Resource Center. [https://lgbt.utah.edu/campus/faculty\\_resources.php](https://lgbt.utah.edu/campus/faculty_resources.php)

English Language Learners: If you are an English language learner, please be aware of several resources on campus that will support you with your language and writing development. These resources include: the Writing Center (<http://writingcenter.utah.edu/>); the Writing Program (<http://writing-program.utah.edu/>); the English Language Institute (<http://continue.utah.edu/eli/>). Please let me know if there is any additional support you would like to discuss for this class.

Undocumented Student Support: Immigration is a complex phenomenon with broad impact—those who are directly affected by it, as well as those who are indirectly affected by their relationships with family members, friends, and loved ones. If your immigration status presents obstacles to engaging in specific activities or fulfilling specific course criteria, confidential arrangements may be requested from the Dream Center. Arrangements with the Dream Center will not jeopardize your student status, your financial aid, or any other part of your residence. The Dream Center offers a wide range of resources to support undocumented students (with and without DACA) as well as students from mixed-status families. To learn more, please contact the Dream Center at 801.213.3697 or visit [dream.utah.edu](http://dream.utah.edu).

Veterans Center: If you are a student veteran, the U of Utah has a Veterans Support Center located in Room 161 in the Olpin Union Building. Hours: M-F 8-5pm. Please visit their website for more information about what support they offer, a list of ongoing events and links to outside resources: <http://veteranscenter.utah.edu/>. Please also let me know if you need any additional support in this class for any reason.

Wellness Statement: Personal concerns such as stress, anxiety, relationship difficulties, depression, cross-cultural differences, etc., can interfere with a student's ability to succeed and thrive at the University of Utah. For helpful resources contact the Center for Student Wellness at [www.wellness.utah.edu](http://www.wellness.utah.edu) or 801-581-7776.

Student Success Advocates: The mission of Student Success Advocates is to support students in making the most of their University of Utah experience ([ssa.utah.edu](http://ssa.utah.edu)). They

can assist with mentoring, resources, etc. Any student who faces challenges securing their food or housing and believes this may affect their performance in the course is urged to contact a Student Success Advocate for support (<https://asuu.utah.edu/displaced-students>).

The Americans with Disabilities Act:

The University of Utah seeks to provide equal access to its programs, services and activities for people with disabilities. If you will need accommodations in the class, reasonable prior notice needs to be given to the Center for Disability & Access, 162 Olpin Union Building, 801-581-5020. CDA will work with you and the instructor to make arrangements for accommodations. All written information in this course can be made available in alternative format with prior notification to the Center for Disability & Access.

Addressing Sexual Misconduct: Title IX makes it clear that violence and harassment based on sex and gender (which includes sexual orientation and gender identity/expression) is a Civil Rights offense subject to the same kinds of accountability and the same kinds of support applied to offenses against other protected categories such as race, national origin, color, religion, age, status as a person with a disability, veteran's status or genetic information. If you or someone you know has been harassed or assaulted on the basis of your sex, including sexual orientation or gender identity/expression, you are encouraged to report it to the University's Title IX Coordinator; Director, Office of Equal Opportunity and Affirmative Action, 135 Park Building, 801-581-8365, or to the Office of the Dean of Students, 270 Union Building, 801-581-7066. For support and confidential consultation, contact the Center for Student Wellness, 426 SSB, 801-581-7776. To report to police, contact the Department of Public Safety, 801-585-2677(COPS).

Campus Safety: The University of Utah values the safety of all campus community members. To report suspicious activity or to request a courtesy escort, call campus police at 801-585-COPS (801-585-2677). You will receive important emergency alerts and safety messages regarding campus safety via text message. For more information regarding safety and to view available training resources, including helpful videos, visit [Safeu.utah.edu](http://Safeu.utah.edu)

University Counseling Center: The University Counseling Center (UCC) provides developmental, preventive, and therapeutic services and programs that promote the intellectual, emotional, cultural, and social development of University of Utah students. They advocate a philosophy of acceptance, compassion, and support for those they serve, as well as for each other. They aspire to respect cultural, individual and role differences as they continually work toward creating a safe and affirming climate for

individuals of all ages, cultures, ethnicities, genders, gender identities, languages, mental and physical abilities, national origins, races, religions, sexual orientations, sizes and socioeconomic statuses.

Office of the Dean of Students: The Office of the Dean of Students is dedicated to being a resource to students through support, advocacy, involvement, and accountability. It serves as a support for students facing challenges to their success as students, and assists with the interpretation of University policy and regulations. Please consider reaching out to the Office of Dean of Students for any questions, issues and concerns. 200 South Central Campus Dr., Suite 270. Monday-Friday 8 am-5 pm.

# Math 1320 Full Course Knowledge Checklist and Practice

## Chapter 6

1. (6.4) Arc Length. Given a curve in the  $x$ - $y$  plane, be able to set up and compute an integral representing the arc length of the curve.

(a) Given a parametric curve  $(x(t), y(t))$  for  $t$  in  $[a, b]$ , commit to memory the arc length formula

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt \quad (1, 2)$$

(b) Given a curve defined by a function  $y = f(x)$  on  $[a, b]$ , know the arc length formula to be

$$L = \int_a^b \sqrt{1 + (f'(x))^2} dx \quad (3)$$

2. (6.5) Average value of a function  $f(x)$ , and the mean value theorem for integrals.

(a) Given a function  $f(x)$  on  $[a, b]$ , commit to memory average value formula

$$f_{avg} = \frac{1}{b-a} \int_a^b f(x) dx \quad (4)$$

and be able to use it in computations.

(b) Commit to memory mean value theorem for integrals (MVTI): If  $f(x)$  is continuous on  $[a, b]$ , then there is a  $c$  in  $[a, b]$  such that

$$f(c) = f_{avg} = \frac{1}{b-a} \int_a^b f(x) dx.$$

- (c) Be able to both check the hypotheses of the MVTI and find a  $c$ -value when required. (b)  
 (d) Understand the geometric perspective of the MVTI, that the area under the curve of  $f(x)$  can be represented by an equivalent-area rectangle with base  $b - a$  and height  $f(c) = f_{avg}$ . (c) (e) Let  $f_{max}$  and  $f_{min}$  be the maxima and minima values of  $f(x)$  on  $[a, b]$ . Know that  $f_{min} \leq f_{avg} \leq f_{max}$ .

3. (6.6) Applications of integration: work and energy.

(a) The general identity:  $W = F d$ , when the force is constant.

(b) Given a physical force function  $f(u)$  on  $[a, b]$ , the energy or work  $W$  done by the force over a distance  $b - a$  is  $\int_a^b f(u) du$  given by

$$\int_a^b f(u) du.$$

$W =$

(c) Given a description of a physical system involving motion over space  $[a, b]$ , students should be able to formulate an appropriate force function that describes the system utilizing common physical laws – Newton's laws, gravitation, electrostatics, and so on; and be able to construct the correct integral that calculates the energy/work done by the system.

(d) Specific force relations include, but are not limited to:

(i) The Newton's second law:  $F = ma(t)$ , where  $x(t)$  is position,  $a(t) = x''(t)$  is acceleration, and  $m$  is the mass of the object.

(ii) Near earth gravitation:  $F = mg$ , where  $m$  is the mass of the object that could vary as a function of time or position – e.g. pulling up leaky buckets, pumping water out of tanks of various shapes. (6,7)

(iii) General gravitation  $F = G \frac{m_1 m_2}{r^2}$ , where  $G$  is a constant. (8) masses  $m_1$  and  $m_2$ ,  
 $r^2$ , where  $r$  is the linear distance between

(iv) Electrostatic force between charges  $q_1$  and  $q_2$ :  $F = k \frac{q_1 q_2}{r^2}$   
 $k$  is a constant.  
 $r^2$ , where  $r$  is the linear distance and

(v) Hookean springs:  $F = -kx$ , where  $k$  is the spring constant. (9)

4. (6.6) Applications of integration: hydrostatic pressure

(a) Know the definition of pressure as a force per unit of area  $P = \frac{F}{A}$ , and to calculate a force on a given surface and pressure, one needs to multiply the area by the pressure –  $F = P A$ .

(b) Know that liquids with density  $\rho$  in a container under the force of near-earth gravity exert pressure on the walls of the container equal to  $P = \rho g d$ , where  $d$  is the depth below the fill-line – i.e., pressure increases directly with depth.

(c) The wall of a filled tank with length  $\ell(y)$  as a function of depth  $y$  experiences a total force constructed from the areas  $dA = \ell(y) dy$  of segments at each  $dy$  depth, each with force  $dF = \rho g y \ell(y) dy$ . The total force is then  $F = \int_0^d \rho g y \ell(y) dy$ .

(d) Based on the tank description, be able to construct the length function  $\ell(y)$  as a function of height  $y$  and use it to compute the total force on a tank wall. (10)

(e) Naturally, the total force on all walls of the vessel must equal the weight of the liquid  $g\rho V$ .

5. (6.6) Applications of integration: center of mass, moments.

(a) If given a density of a linear object  $\lambda(x)$  (mass/length) laid on the  $x$ -axis on  $[a, b]$

(i) The total mass:  $m = \int_a^b \lambda(x) dx$ .

(ii) The center of mass, a position in space  $\bar{x} = \frac{1}{m} \int_a^b x \lambda(x) dx$  representing the balance point of the object.



## Chapter 8

### 6. (8.1) Sequences

(a) Know the definition of a sequence as an infinite-sized ordered list of numbers  $\{a_n\}_{n=1}^{\infty}$

$$_{n=1}^{\infty} = \{a_1, a_2, a_3, \dots\}.$$

A sequence is indexed by a whole number  $n$  that indicates the point in the

sequence. Page 2

(b) Sequences can be any list of numbers, but there are several types that are of specific interest: (11)

(i) Functions of  $n$ :  $a_n = f(n)$  defined by a specified mathematical operation on an input  $n$ ; e.g.,  $a_n = 1/n^2$ , or  $a_n = e^{-n} \sin(n)$ .

(ii) Recursively defined sequences:  $a_{n+1}$  is defined in terms of  $a_n$ ; e.g.,  $a_1 = 2$  and  $a_{n+1} = 0.5(3 - a_n)$ .

(iii) Geometric: given  $a, r$ , define  $a_n = ar^n$ .

(iv) Alternating sign:  $(-1)^n$  or  $(-1)^{n-1}$ .

(c) Be able to recognize when two differently-indexed or represented sequences are equivalent, or not. (12)

(d) Know the definition of convergence for sequences: If  $\lim_{n \rightarrow \infty} a_n = L$  exists, then the sequence is convergent.

(e) Be able to compute limits of sequences when possible; particularly, if  $a_n = f(n)$ , and  $f(x)$  is an indeterminate form, you can apply l'Hospital's rule when needed.

(f) Know how to utilize the Monotone Sequence Theorem (MST): If the sequence is monotonic in creasing/decreasing, and the sequence is bounded above/below, respectively, then the sequence is convergent. When applying the MST be sure to verify/assert/prove that the hypotheses of the theorem are met in order to assert convergence.

### 7. (8.2) Series

(a) An *infinite series*  $s$  is a  $\sum_{n=1}^{\infty} a_n$ .

sum of a sequence:  $s = \sum_{n=1}^{\infty} a_n$

(b) A *partial sum*  $s_N$  is the sum of the first  $N$  terms:  $s_N = \sum_{n=1}^N a_n$

(c) Know the definition of convergence and divergence of a series:  $s$  converges when the sequence of partial sums converges, that is  $\lim_{N \rightarrow \infty} s_N = s$  is a finite number;  $s$  diverges when it diverges to infinity. (13)  $\lim_{N \rightarrow \infty} s_N$  is undefined or

(d) Be able to recognize special categories of series and their rules for convergence/divergence:

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$ar^n$  converges when  $|r| < 1$ , equal to  $\frac{a}{1-r}$ . (14)

(ii)  $p$ -series:  $\sum_{n=1}^{\infty} \frac{1}{n^p}$   $p$  any number,  $s = X^{\infty}$

integral test (see below).

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$n^p$  converges when  $p > 1$  and diverges when  $p \leq 1$ , by the

(e) Know the basic series manipulation identities. If  $\sum a_n$  and  $\sum b_n$  are convergent and  $c$  any number, then (15)

(i)  $\sum (a_n \pm b_n) = \sum a_n \pm \sum b_n$  is convergent.

(ii)  $\sum ca_n = c \sum a_n$ .

8. (8.2-4) Series convergence tests  $a_n$ , where  $h$  is some integer.

for  $s = \sum_{n=h}^{\infty} a_n$

(a) Test for divergence: A series is divergent if  $\lim_{n \rightarrow \infty} a_n \neq 0$  or undefined. (16) (b) Integral

test: If  $a_n = f(n)$ , and  $f(x) > 0$  for  $x$  in  $[h, \infty)$ , and  $f(x)$  is decreasing, then  $\sum_{n=h}^{\infty} a_n$  has the same convergence/divergence property as  $\int_h^{\infty} f(x) dx$  as  $s$ . (17)

(c) Comparison test (bounding by a known series): Suppose  $a_n, b_n > 0$ , and suppose  $\sum b_n$  is known to converge, and if  $a_n \leq b_n$  then  $s$  converges; conversely, if  $\sum b_n$  is known to diverge, and if  $a_n \geq b_n$  then  $s$  diverges. (18)

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(d) Limit comparison test: suppose  $\sum b_n$  is known to converge or diverge, and  $a_n, b_n > 0$ . If  $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = c > 0$ , where  $c$  is finite, then  $s$  shares the same convergence property. (19)

(e) Alternating series test: Suppose  $a_n = (-1)^{n-1} b_n$ , where  $b_n > 0$ ,  $b_{n+1} \leq b_n$ , and  $\lim_{n \rightarrow \infty} b_n = 0$ , then  $s$  converges. (20)

(f) Know the definition of absolute convergence:  $\sum a_n$  converges absolutely if  $\sum |a_n|$  is convergent. An absolutely convergent series is convergent, but a convergent series does not necessarily converge absolutely.

Compute  $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = L$ . The scenarios are

(g) The ratio test: as follows (21)

- (i) If  $L < 1$  the series is absolutely convergent.
- (ii) If  $L > 1$  or is  $= \infty$ , then the series is divergent. (iii)
- If  $L = 1$  or DNE, then the test is indeterminate.

9. (8.2-4) Series estimation methods for convergent  $s = \sum_{n=1}^{\infty} a_n$

(a) A geometric series does not need to be estimated, the exact value is known:  $s = \frac{a}{1-r}$  if  $|r| < 1$ .

(b) Students should understand the need for series estimation methods. Any *partial sum*  $s_N$  can be used as an approximation, but one might want to know how accurate each  $s_N$  is – estimation techniques give information as to how close a given partial sum  $s_N$  is to  $s$ .

(c) Know the definition of a remainder  $R_N = s - s_N$  value. (d) The remainder estimate for integral tests  $a_n = f(n) > 0$ :  $R_N = \int_N^{\infty} f(x) dx$ .

The remainder can be considered

$R_N = \int_N^{\infty} f(x) dx$  the error between an approximation and the exact

$$\int_N^{\infty} f(x) dx \leq R_N \leq \int_{N+1}^{\infty} f(x) dx \quad (22)$$

$\int_N^{\infty} f(x) dx$ .

(e) Alternating series estimation theorem: if  $s = \sum_{n=1}^{\infty} (-1)^{n-1} b_n$  series test hypotheses, then  $|R_N| \leq b_{N+1}$ .

where  $b_n$  satisfies the alternating

(23) 10. (8.5-6) Power series

power series in  $x$ :  $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ .

(a) Know the definition of a function defined by a

(b) Be able to use convergence tests, principally the ratio test, to determine for which values  $x$  does the series converge.

(c) Know the definition of the radius and interval of convergence and be able to determine them for a given series. (24)

(d) Using the known representation of a geometric power series by  $\sum_{n=1}^{\infty} a/(1-r)$ , be able to construct other power series by algebraic manipulation, integration, and differentiation. (25, 26, 27)

(e) Know under which conditions a function  $f(x)$

represented by a power series  $\sum_{n=0}^{\infty} c_n(x-a)^n$  is

differentiable ( $f'(x)$  exists) and computable by

differentiation of the series term by term  $\sum_{n=0}^{\infty}$

$nc_n(x-a)^{n-1}$ .

$a)^{n-1}$ .

(f) Know under which conditions a function  $f(x)$

represented by a power series  $\sum_{n=0}^{\infty} c_n(x-a)^n$  is integrable

over  $R$  ( $\int f(x)dx$  exists) and computable by

integration of the series term by term  $\sum_{n=0}^{\infty} C + X$

11. (8.7-8) Taylor and Maclaurin series

$c_n(x-a)^{n+1} / (n+1)$ .

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(a) Be able to determine/compute the special power series representation of a function near a point  $x = a$  – termed a Taylor series – as

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n, \text{ for } |x-a| < R.$$

This is known as a *Taylor expansion of  $f$  at  $x = a$* . When  $a = 0$ , the Taylor series is termed a Maclaurin series. (28, b)

(b) Know the definition of the  $N$ th partial sum as the order- $N$  Taylor expansion

$$T_N(x) = \sum_{n=0}^N \frac{f^{(n)}(a)}{n!} (x-a)^n$$

(c) Know how to use the Taylor's Inequality to determine the bound on the error  $R_N(x)$  in a neighborhood of  $x = a$ :  $|x - a| \leq d$ : If  $|f^{(N+1)}(x)| < M$  – the bound on the  $(N + 1)$ th derivative for points  $|x - a| \leq d$ , then the remainder/error between  $T_N(x)$  and  $f(x)$  is within the range (d, 30, 31)

$$|f(x) - T_N(x)| = |R_N(x)| \leq M$$

$$\frac{M}{(N+1)!} |x-a|^{N+1}, \text{ for } |x-a| \leq d.$$

- (d) If given  $f(x)$  and a required error tolerance  $\geq |R_N(x)|$  on  $|x - a| \leq d$ , students should be able to find the correct Taylor polynomial of order  $N$  that ensures this error tolerance is met; conversely, given an  $N$ , students should be able to determine the maximal error bound of  $|R_N(x)|$  on  $|x - a| \leq d$ . (32)

## Chapter 9

### 12. (9.1) 3D coordinate systems

- (a) Know how to compute distances between points in the Cartesian 3D coordinate system.  
 (b) Know how to represent a set of points forming a sphere as an equation of equal distance  $r$  from a center point  $(x_0, y_0, z_0)$  to the sphere surface:

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2.$$

- (c) Know how to represent a set of points forming a cylinder as an equation of equal distance  $r$  from the vertical axis located at  $(x_0, y_0)$ :

$$(x - x_0)^2 + (y - y_0)^2 = r^2, \text{ where } z \in \mathbb{R}.$$

- (d) If given an equation of a sphere or a cylinder, be able to identify the center point or axis, respectively, and the radius. (33)  
 (e) Be able to write out in set notation the points lying in a given surface. (34)

### 13. (9.2) Vectors in 2- and 3D space

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- (a) Know how to compute the algebraic vector computations of vector addition, subtraction, and scalar multiplication using the algebraic properties on page 643. (35)  
 (b) If given graphical depictions of two or more vectors, be able how to graphically represent vector addition, subtraction, and scalar multiplication. (36)  
 (c) Given a vector  $\sim u$ , be able to produce a unit-length vector  $\tilde{U}$  in the same direction  $\tilde{U} = \frac{\sim u}{|\sim u|}$ ; also be familiar with the standard unit vectors  $\tilde{i}$ ,  $\tilde{j}$ , and  $\tilde{k}$ . (37)  
 (d) Be able to recognize and use various vector notations of angle brackets  $\sim u = hu_1, u_2, u_2i$ , and using standard unit vectors  $\sim u = u_1\tilde{i} + u_2\tilde{j} + u_3\tilde{k}$ .  
 (e) Be able to compute the magnitude of a vector using the formula  $|\sim u| = \sqrt{u_1^2 + u_2^2 + u_3^2}$ .

### 14. (9.3) Dot product

- (a) Know how to compute the dot product between two vectors

$$\sim u \cdot \sim v = u_1v_1 + u_2v_2 + u_3v_3.$$

(38)

- (b) Be able to use the dot product angle-formula:

$$\sim u \cdot \sim v = |\sim u||\sim v| \cos(\theta).$$

(39)

(c) Know how to identify the dot product number in relation to the angle  $\theta$  between the vectors when depicted graphically.

(d) Given a force vector  $\vec{F}$  on an object moving on a path given by  $\vec{d}$ , students should be able to compute the work  $W$  done on the object as

$$W = \vec{F} \cdot \vec{d}.$$

(41)

(e) Be able to compute the angle between vectors using the dot product-based formula

$$\theta = \cos^{-1} \frac{\vec{u} \cdot \vec{v}}{|\vec{u}||\vec{v}|}$$

(39)

(f) Know that two vectors are orthogonal when  $\vec{u} \cdot \vec{v} = 0$ . (41)

(g) Be able to perform vector calculations using the algebraic properties of the dot product shown on page 651. (42)

(h) Be able to find the scalar component of vector  $\vec{u}$  in the direction of vector  $\vec{v}$  using the *comp*

$$\text{formula } \text{comp}_{\vec{v}}(\vec{u}) = \frac{\vec{u} \cdot \vec{v}}{|\vec{v}|}$$

$$|\vec{v}|$$

(i) Be able to graphically depict the quantity  $\text{comp}_{\vec{v}}(\vec{u})$  and explain how it relates to the vectors.

(43) (j) Be able to find the vector projection of vector  $\vec{u}$  in the direction of vector  $\vec{v}$  using the *proj* formula

$$\text{proj}_{\vec{v}}(\vec{u}) = \text{comp}_{\vec{v}}(\vec{u}) \frac{\vec{v}}{|\vec{v}|} = \frac{\vec{u} \cdot \vec{v}}{|\vec{v}|^2} \vec{v}.$$

(44)

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(k) Be able to graphically depict the vector projection as the vector of length  $\text{comp}_{\vec{v}}(\vec{u})$  in the direction of  $\vec{v}$ .

#### 15. (9.4) Cross product

(a) Students should be able to compute cross product between two vectors to

$$\text{be } \vec{u} \times \vec{v} = hu_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1i.$$

(45)

(b) Students should be able to utilize all the algebraic properties of the cross product (page 656) to compute various vector calculations. (46)

- (c) The magnitude of the cross product relates to the angle between vectors according to the torque formula in (a)

$$|\sim u \times \sim v| = |\sim u||\sim v|\sin(\theta),$$

and is equal to the area of the parallelogram formed by the  $\sim u$  and  $\sim v$  sides. (47)

- (d) Students should be able to infer the properties of the cross product for two special  $\sim u$  and  $\sim v$  relationships.

(i)  $\sim u \times \sim v$  is orthogonal to both  $\sim u$  and  $\sim v$ .

(ii) If  $\sim u$  and  $\sim v$  point along the same direction—either  $\theta = 0$  or  $\theta = \pi$ —then  $\sim u \times \sim v = \sim 0$ . (e)

Students should be able to compute torque: If a vector force  $\sim F$  is applied to a lever arm  $\sim r$  to effect a twisting force on an axis passing through the origin (the base of  $\sim r$ ), the torque  $\tau$  is the force vector that points along the axis of twisting, and has magnitude

$$|\sim \tau| = |\sim F| |\sim r| \sin(\theta),$$

where  $\theta$  is the angle between  $\sim F$  and  $\sim r$ . The direction of  $\sim \tau$  is chosen with the right hand rule. (48)

- (f) Students should be able to compute the volume  $V$  of a parallelepiped formed by three vectors  $\sim u$ ,  $\sim v$ , and  $\sim w$  using the triple product formula

$$V = \sim w \cdot (\sim u \times \sim v).$$

(49)

#### 16. (9.5) Equations for lines and planes

- (a) Students should commit to memory the two equation forms for lines and be able to be able to produce one from the other:

(i) Symmetric:  $\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}$

(ii) Parametric:  $\sim r(t) = \sim vt + \sim r_0, t \in \mathbb{R}$ .

(50)

- (b) Students should commit to memory the three equation forms for planes and be able to be able to produce one from the others:

(i) Normal form:  $\sim n \cdot (\sim r - \sim r_0) = 0$

(ii) Scalar:  $ax + by + cz = d$ .

(iii) Parametric:  $\sim r(s, t) = \sim us + \sim vt + \sim r_0, s, t \in \mathbb{R}$ .

(51)

- (c) Students should be able to produce the correct equation for a line or plane if given two, or three points respectively, that reside in the line/plane. (51)

- (d) Given two objects, either points, lines, or plane objects, students should be able to the object defined by their intersection (if it exists). (52)

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- (e) Students should be able to find the angle of intersection between two line or plane objects. (53)

- (f) Students should be able to find the distance between a plane or line and a point in space by finding the closest-distance point. (55)

(g) Given a line or a plane, students should be able to find orthogonal or parallel lines or planes, possibly subject to additional conditions. (56,57)

17. (9.6) Multivariate functions and surfaces

(a) Be able to determine the domain and range of a multivariable function  $z = f(x, y)$  and express the domain as a 2D set.

(b) Be able to draw or graphically depict the rudiments of the surface generated from a multivariable function  $z = f(x, y)$ . This can be done several strategies, when applicable.

(i) Holding one variable fixed (e.g.,  $y$ ) and drawing cross sections in the other variable (e.g.,  $x$ ).

(ii) Using special geometric principles if applicable (e.g., distance formulas, etc). (iii) Finding level sets: fix  $z$  and draw a contour lines in the  $x$ - $y$  plane.

(c) Be familiar with certain prototypical surfaces and their corresponding functions: Paraboloids, hyperbolic paraboloids, ellipsoids, cones, ... (see page 679).

(58)

18. (9.7) Cylindrical and spherical coordinates

(a) Students should be able to convert between cartesian coordinates and cylindrical coordinates:

$$x = r \cos(\theta), y = r \sin(\theta), \\ \tan(\theta) = \frac{y}{x}, r^2 = x^2 + y^2, z = z.$$

(59)

(b) Students should be able to convert between cartesian coordinates and spherical coordinates:

$$x = \rho \sin(\varphi) \cos(\theta), y = \rho \sin(\varphi) \sin(\theta), z = \rho \cos(\varphi) \\ \tan(\theta) = \frac{y}{x}, \rho^2 = x^2 + y^2 + z^2, r = \rho \sin(\varphi).$$

(59)

(c) Be able to identify basic surfaces defined by equations expressed in these coordinate systems, like cones, spheres, etc. (60)

## Chapter 10

19. (10.1) Vector functions

(a) Students should be able to graphically represent space curves given by vector functions of the

$$\text{form } \vec{r}(t) = hx(t), y(t), z(t), t \in [a, b].$$

(61)

(b) Given a description of a basic space curve, students should be able to formulate a vector function  $\vec{r}(t)$  that represents the curve.(62)

(c) Be able to formulate a space curve as a vector function  $\vec{r}(t)$  if the space curve is defined by a set of equations. (63)

20. (10.2) Derivatives and integrals of vector functions.



(a) Students should be able to compute derivatives of vector functions

$$\tilde{r}'(t) = \langle \dot{x}(t), \dot{y}(t), \dot{z}(t) \rangle, t \in [a, b].$$

(64)

(b) Students should be able to graphically depict the direction and magnitude of  $\tilde{r}'(t)$  as tangent to the space curve with magnitude equal to the speed  $v(t) = |\tilde{r}'(t)|$ .

(c) Students should be able to compute derivatives of vector expressions using the differentiation rules on page 704. (65)

(d) Students should be able to compute integrals of vector functions

$$\int_a^b \tilde{r}(t) dt = \int_a^b \langle x(t), y(t), z(t) \rangle dt.$$

## 21. (10.3) Arc length and curvature

(a) Students should be able to produce and compute the arc length  $L$  formula of a space curve, defined as

$$L = \int_a^b |\tilde{r}'(t)| dt. \quad \text{where } ds = |\tilde{r}'(t)| dt. \quad (66,67)$$

(b) Know how the graphical meaning and how to compute the tangent vector:  $\tilde{T}(t) = \frac{\tilde{r}'(t)}{|\tilde{r}'(t)|}$ .

(c) Students should be able to explain and compute the curvature  $\kappa(t)$  of a space curve as the magnitude of rate of change of  $\tilde{T}$  with respect to arc

$$\text{length } ds: \kappa = \frac{d\tilde{T}}{ds}$$

computing curvature are

$$\kappa(t) = \frac{|\tilde{T}'(t)|}{|\tilde{r}'(t)|}$$

$$|\tilde{r}'(t)| = |\tilde{r}'(t) \times \tilde{r}''(t)|$$

$$|\tilde{r}'(t)|^3,$$

$$\text{where } ds = |\tilde{r}'(t)| dt. \quad (68)$$

. The constructive formulas for

(d) Students should know how to compute the, tangent, normal and binormal vectors forming a local coordinate system  $(\tilde{T}(t), \tilde{N}(t), \tilde{B}(t))$  at every point  $\tilde{r}(t)$  of a space curve (see page 712-713. Students should be able to graphically depict the directions of these vectors where appropriate. (69)

(e) Know the definitions of the normal and osculating planes of a space curve, know their normal vectors and how to find equations of the planes. Be able to describe the behavior of these planes as a function of time  $t$ .

## 22. (10.4) Motion in space: velocity and acceleration

- (a) Students should be able to compute the velocity and acceleration vectors given a position vector function  $\vec{r}(t)$

$$\vec{v}(t) = \vec{r}'(t), \vec{a}(t) = \vec{v}'(t) = \vec{r}''(t).$$

- (b) Students should be able to decompose the velocity vector into its direction unit vector  $\vec{T}(t)$ , and magnitude given by speed  $v(t) = |\vec{v}(t)|$ :

$$\vec{v}(t) = v(t)\vec{T}(t)$$

- (c) Students should be able to decompose the acceleration vector into its tangential component  $T$  (fore/aft direction), and its normal component  $N$  (side-to-side, or steering direction) as expressed

as: (t), Page 9

$$\vec{a}(t) = v^0(t)\vec{T}(t) + \kappa(t)v^2(t)\vec{N}(t)$$

where  $v^0(t)$  is the tangential component of acceleration

$$v^0(t) = a_T(t) = \vec{T}(t) \cdot \vec{a}(t),$$

and  $\kappa(t)v^2(t)$  is the normal component of acceleration:

$$\kappa(t)v^2(t) = a_N(t) = \vec{N}(t) \cdot \vec{a}(t).$$

(70)

- (d) Given a vector  $\vec{x} = hx_1, x_2, x_3j$  in the standard coordinate system  $(\vec{i}, \vec{j}, \vec{k})$ , be able to represent  $\vec{x}$  in an alternative coordinate system defined by orthogonal vectors  $\vec{u}, \vec{v}, \vec{w}$  as

$$\vec{x} = a\vec{u} + b\vec{v} + c\vec{w}$$

where the triplet  $(a, b, c)$  are the coordinates for  $\vec{x}$  in the new system given by

$$a = \frac{\vec{x} \cdot \vec{u}}{|\vec{u}|^2}, b = \frac{\vec{x} \cdot \vec{v}}{|\vec{v}|^2}, c = \frac{\vec{x} \cdot \vec{w}}{|\vec{w}|^2}.$$

(71)

## Chapter 11

23. (11.1) Functions of several variables  $f(x, y) = z$ .

- (a) Students should be able to determine the domain and range of multivariate functions.

- (b) Students should be able to determine the nature of the relationship between points  $(x, y)$  and  $z$  using a methodical approach involving the synthesis of multiple approaches:
- (i) Forming a table of values for a relevant range of example points  $(x, y, z)$ .
  - (ii) Studying the algebraic relationship between  $(x, y)$  and  $z$ .
  - (iii) Form graphical depictions of the surface by setting a variable equal to constants and drawing the resulting traces or level curves of the remaining variables. Note that for each trace formed at each  $(x, y)$  point, the traces must intersect for the surface to be defined by a function. (72,73,74)
- (c) The aforementioned items must also be performed on any 3D coordinate system: Euclidian, cylindrical, spherical.

24. (11.2) Limits and continuity of  $f(x, y) = z$ .

- (a) Students should understand that  $\lim_{(x,y) \rightarrow (a,b)} f(x, y) = L$  means that  $f(x, y)$  takes on  $z$ -values that are exceedingly close to  $L$  whenever  $(x, y)$  is near  $(a, b)$ . This means that any trajectory  $(x, y) \rightarrow (a, b)$  yields  $z \rightarrow L$ .
- (b) Students should be able to craft a clear argument that a given limit *does not exist* by taking two limits  $(x, y) \rightarrow (a, b)$  along two distinct paths  $C_1$  and  $C_2$  and establishing that the limits (if they exist) are different. If the limits on the two paths are the same, then no conclusions can be made about the existence of the limit. (75,76)
- (c) Continuity definition: Students should know the definition of continuity and use it to assert existence of a limit. Definition: If a limit exists at a point  $\lim_{(x,y) \rightarrow (a,b)} f(x, y) = L$ , and  $f(a, b) = L$ , then the function is continuous at  $(a, b)$ . Limit existence: If a function is known to be continuous, students should be able to use this fact to assert what the limit is ( $L$ ).

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- (d) Determining continuity: Just as with single-variable functions, certain categories of functions like polynomials, trigonometrics, logs, exponentials, and function compositions, sums, products, and divisions of these functions, are continuous, if not at singular points (e.g., division by zero points or other singularities). That is, if a function is in such a category and non singular at the point in question, the function is continuous at that point. Students should be able to identify these functions and be able to assert continuity when needed. (77)

25. (11.3) Partial derivatives

- (a) Students should be able to compute partial derivatives using the limit-based definition.

$$\frac{\partial f}{\partial x}(a, b) = \lim_{h \rightarrow 0} \frac{f(a+h, b) - f(a, b)}{h}, \quad \frac{\partial f}{\partial y}(a, b) = \lim_{h \rightarrow 0} \frac{f(a, b+h) - f(a, b)}{h}$$

- (b) Students should be able to compute partial derivatives using standard differentiation rules. (78,79) (c) Students should be familiar with the various partial derivative notations

$$\frac{\partial f}{\partial x} = f_x = \partial_x f = D_x f, \quad \frac{\partial f}{\partial y} = f_y = \partial_y f = D_y f,$$

and second partial derivative notations, for instance

$$\frac{\partial^2 f}{\partial x^2} = f_{xx} \partial_{xx} f = D_{xx} f, \quad \frac{\partial^2 f}{\partial x \partial y} = f_{xy} \partial_{xy} f = D_{xy} f$$

$$\frac{\partial^2 f}{\partial x \partial y} = (f_y)_x = f_{yx} = \frac{\partial}{\partial y} f_x = D_{yx} f.$$

- (d) Given a differentiable function (see below), students should be able to determine the slope of a function's tangent lines in the  $x$  or  $y$  directions using partial derivatives and be able to use this information to understand the local properties of the surface of  $f$ .
- (e) Students should be able to use Clairaut's theorem to assert the equality of mixed partials: If  $f$  is defined in a region around  $(a, b)$  and the mixed partials  $f_{xy}$  and  $f_{yx}$  are both continuous there (i.e. the mixed partials smooth), then  $f_{xy} = f_{yx}$  at  $(a, b)$ .

26. (11.4) Tangent planes and linear approximations

- (a) Students should be able to compute/find tangent plane of a surface defined by  $f(x, y) = z$  at a point  $(x_0, y_0, z_0)$  is defined by

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0),$$

where the normal vector to the tangent plane is  $\vec{n} = \langle f_x(x_0, y_0), f_y(x_0, y_0), -1 \rangle$ . (80) (b) Students should be able to use the tangent plane of function at a point  $(x_0, y_0, z_0)$  as a way to approximate  $f(x, y)$  at points  $(x, y)$  near  $(x_0, y_0)$ :

$$f(x, y) = z \approx z_0 + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

(81)

- (c) Differentiability definition:  $f$  is differentiable at a point  $(a, b)$  if the error  $R(x, y)$  between a function and its linear approximation goes to zero as  $(x, y) \rightarrow (a, b)$  in such a manner that  $R(x, y)/(x-a) \rightarrow 0$  and  $R(x, y)/(y-b) \rightarrow 0$ . That is, a tangent plane well approximates the surface in a way that  $R$  goes to zero faster than linearly (e.g., quadratically). This is a difficult definition put here for reference only.
- (d) Students should be able to use the differentiability theorem—If the partials of  $f$  exist and are continuous at  $(a, b)$ , then  $f$  is differentiable—to determine differentiability and the usability of a tangent plane to make good approximations.

27. (11.5) Chain rule

- (a) Given a differentiable trajectory in 2D  $\vec{r}(t) = hx(t), y(t)i$ , students should be able to compute the derivative of  $f(x, y) = z$  with respect to time on the trajectory using the chain rule in 2D:

$$\frac{dz}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt}$$

(82)

- (b) Student should be able to compute derivatives similarly defined for functions of three variables  $u = f(x, y, z)$  defined on 3D trajectories  $\vec{r}(t) = hx(t), y(t), z(t)i$ :

$$du$$

$$dt = f_x x^0 + f_y y^0 + f_z z^0.$$

(83)

- (c) Students should be able to graphically depict and/or describe  $z(t)$  and its surrounding local surface based on information in  $z^0(t)$  computed as above in (a). (84)

- (d) Implicit functions: Given an equation  $F(x, y) = 0$  that defines implicitly a function  $f(x) = y(x) = y$ , students should be able to compute the derivative of  $f(x) = y$  using the chain rule:

$$= 0 = \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy \implies \frac{dy}{dx} = -\frac{F_x}{F_y}$$

28. (11.6) Directional derivatives and gradient vector.

- (a) Students should be able to compute the directional derivative of a differentiable function  $f(x, y) = z$  at a point  $(x, y)$  in any direction  $\vec{u} = ha, bi = h\cos(\theta), \sin(\theta)i$  defined as a unit vector  $|\vec{u}| = 1$  given by the formula

$$D_{\vec{u}}f(x, y) = f_x(x, y)a + f_y(x, y)b.$$

(85)

- (b) Students should be able to graphically identify  $D_{\vec{u}}f(x, y)$  as the slope of the tangent line on the surface of  $f$  in the direction  $\vec{u}$ .
- (c) Students should know how to compute the gradient vector  $\nabla f = hf_x, f_yi$
- (d) Students should be able identify direction of travel with the steepest ascent of the surface is defined by the maximal  $D_{\vec{u}}f$ , which will always be in the direction of  $\nabla f$ . Moreover,  $\nabla f$  will be orthogonal to any level curve tangent vector. (86)
- (e) Students should be able to express directional derivative by the gradient vector as  $D_{\vec{u}}f(x, y) = \nabla f \cdot \vec{u}$ .
- (f) Given a surface defined by the equation  $F(x, y, z) = k$ , students should be able to find the normal vector of the tangent plane of the surface at a point  $(x_0, y_0, z_0)$  as  $\vec{n}(x_0, y_0, z_0) = \nabla F(x_0, y_0, z_0)$ , with tangent plane equation given by

$$0 = \vec{n}(x_0, y_0, z_0) \cdot hx - x_0, y - y_0, z - z_0i.$$

(87)

29. (11.7) Minimum and maximum values of a surface

- (a) Students should utilize Fermat's theorem (not his last theorem) to aid in finding local min and max values: if  $f$  has a local min/max, at  $(x, y)$ , then  $\nabla f(x, y) = \vec{0}$ —the point  $(x, y)$  is a critical point if  $\nabla f(x, y) = \vec{0}$ . (a)

(b) Students should be able to use the second derivative test of a differentiable  $f$  to determine if a critical point  $(x, y)$  is a local minimum, maximum, saddle, or indeterminate point: Define

$$D_{(x,y)} = \begin{matrix} f_{xx} f_{yy} - f_{xy}^2 \\ = f_{xx} f_{yy} - f_{xy}^2 \end{matrix}$$

If  $D > 0$  and  $f_{xx} > 0 \Rightarrow$  local min; if  $D > 0$  and  $f_{xx} < 0 \Rightarrow$  local max;  $D < 0 \Rightarrow$  saddle; if  $D = 0 \Rightarrow$  indeterminate. (b, 89)

(c) Students should be able to determine absolute min and max values on a closed set  $U$  by checking all critical points within  $U$  and its boundary points. (90)

30. (11.8) Lagrange multipliers—optimization under constraints.

(a) Students should recognize that many optimization problems in science and engineering involve maximizing/minimizing a given value  $z = f(x, y)$  subject to some given constraint on  $(x, y)$  to satisfy a given equation  $g(x, y) = 0$ —this could also be expressed in three or more independent variables.

(b) Solving the problem in (a) involves solving

$$\nabla f = \lambda \nabla g,$$

for an unknown scalar  $\lambda$  to find critical point  $(x, y)$ . (91, 92)

(c) Students should be able to graphically and mathematically explain the rationale behind (c):  $g(x, y) = 0$  defines a level curve  $\sim r(t)$ . The min/max  $z(t)$  critical points will be attained traveling along on  $\sim r(t)$ :  $z(t) = f(\sim r(t))$ .  $z(t)$  will be a critical point when  $z'(t) = 0$ . By the chain rule  $z'(t) = f_x x' + f_y y' = 0 = \nabla f \cdot \sim r'(t)$ , meaning that the gradient of  $f$  is orthogonal to the level curve only at critical points. Note also that  $0 = \frac{dg}{dt} = \nabla g \cdot \sim r'(t)$ , meaning the gradient of  $g$  is orthogonal to the level curve everywhere. Therefore  $\nabla g$  and  $\nabla f$  must be pointing along the same direction at critical points; or rather, they are scalar multiples of each other, thus verifying the equation in (c) defining the critical points.

## Chapter 12

31. (12.1) Double integrals over rectangular regions.

(a) Be able to visualize functions  $f(x, y) = z$  over rectangular regions  $[a, b] \times [c, d]$ . (b) Understand that the volume under the surface of a function  $f(x, y) = z$  over rectangular region  $R = [a, b] \times [c, d]$  can be computed by a double Riemann sum

$$V \approx \sum_{i=1}^M \sum_{j=1}^N f(x_{ij}^*, y_{ij}^*) \Delta A,$$

where  $R$  is broken up into many rectangular patches  $R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$  with area  $\Delta A = \Delta x \Delta y$ , where

$$x_i = a + i\Delta x, y_j = c + j\Delta y, \Delta x = b - a$$

$$N, \Delta y = \frac{d-c}{M}$$

with sample points  $(x_{ij}^*, y_{ij}^*)$  in  $R_{ij}$ .

(c) Be able to numerically approximate the volume using the double Riemann sum and choosing the sample points by one of the following methods (93)

i. Basic midpoint rule: The sample points are in the middle of the rectangle  $R_{ij}$ , that is  $x_{ij}^* =$

$$x_{i-1} + \frac{\Delta x}{2} \text{ and } y_{ij}^* = y_{j-1} + \frac{\Delta y}{2}$$

$$x_{ij}^* = \frac{x_{i-1} + x_i}{2}$$

2.

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ii. Left-endpoint rule: The sample points are in the lower left corner of the rectangle  $R_{ij}$ , that is

$$x_{ij}^* = x_{i-1} \text{ and } y_{ij}^* = y_{j-1}.$$

iii. Right-endpoint rule: The sample points are in the upper right corner of the rectangle  $R_{ij}$ , that is  $x_{ij}^* = x_i$  and  $y_{ij}^* = y_j$ .

(d) Be able specify the a Riemann sum as a limit of finite sums from above as the definition of a double integral

$$V = \lim_{N \rightarrow \infty, M \rightarrow \infty} \sum_{i=1}^N \sum_{j=1}^M f(x_{ij}^*, y_{ij}^*) \Delta A = \iint_R f(x, y) dA$$

Iterated integrals.

(a) Be able to apply Fubini's theorem: If  $f(x, y) = z$  is continuous on the rectangle  $R = [a, b] \times [c, d]$ , then the double integral can be computed by either of the following iterated integrals:

$$\iint_R f(x, y) dA = \int_c^d \int_a^b f(x, y) dx dy = \int_a^b \int_c^d f(x, y) dy dx,$$

where either iterated integral is computed by first computing the inner integral in one variable, keeping the other variable a fixed parameter, then subsequently computing the outer integral afterwards.

(b) Be able to identify, where relevant, when one iterated integral is more advantageous or computationally simpler than the other. (94)

(c) When the integrand is a product of two functions in one variable, that is  $f(x, y) = g(x)h(y)$ , be able to exploit the identity: (105)

$$\int_c^d \int_a^b g(x)h(y) dx dy = \int_a^b g(x) dx \int_c^d h(y) dy$$

33. (12.3) Integration over general regions.

(a) Be able to specify type 1 and type 2 regions  $D$  in set notation:

$$\text{Type I: } D = \{(x, y) | a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\}$$

$$\text{Type II: } D = \{(x, y) | c \leq y \leq d, h_1(y) \leq x \leq h_2(y)\}$$

(b) Given a set of equations and/or lines defining the boundaries of a region  $D$ , be able find the intersection points for the boundary curves and formulate the region  $D$  as a type I or II region when possible. If the full region is not type I or II, be able to split the region into subsets that are when possible. (98)

(c) Given a type I or II region, be able to express the double integral  $\iint_D f(x, y) dA$  as one of the iterated integrals containing functions of the outer integral variable in limits of integration of the inner integral:

Type I: compute these integrals. (95, 96)

$$\int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y) dy dx$$

Type II:  $\int_c^d \int_{h_2(y)}^{h_1(y)} f(x, y) dx dy$

Be able to

(d) If a region  $D$  is both type I and II, be able to assess which iterated integral is most efficient to compute. (97)

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(e) Be able to compute the area of a region  $D$ , as (101)

$$A = \iint_D 1 dA$$

$D$

(f) If a region  $D$  cannot be written as a region of type I or II, be able to break it up into regions  $D_1$  and  $D_2$  of type I and/or II, that do not overlap, and know

$$\iint_D f(x, y) dA = \iint_{D_1} f(x, y) dA + \iint_{D_2} f(x, y) dA$$

### 34. (12.4) Integration with polar coordinates.

(a) Be able to convert 2D regions expressed in Cartesian coordinates  $(x, y)$  into polar coordinate  $(r, \theta)$  through the relations: (99)

$$x = r \cos(\theta), y = r \sin(\theta), r^2 = x^2 + y^2.$$

(b) Commit to memory the area element in polar coordinates:  $dA = r dr d\theta$ .

(c) Be acquainted with the derivation of the area element formula, and understand the essential principle that the area  $\Delta A$  covered due to an angle change  $\Delta\theta$  and radial change  $\Delta r$  is in direct proportion to the distance  $r$  of the area from the origin. That is, if the area in question is further from the origin, the area in question will be larger than if it were closer to the origin:  $\Delta A \propto r$ .

(d) Be able to compute a double integral over a region  $D = \{(r, \theta) | a \leq r \leq b, \alpha \leq \theta \leq \beta\}$  by transforming it into an iterated integral in polar coordinates:

$$\iint_D f(x, y) dA = \int_{\beta}^{\alpha} \int_a^b f(r \cos(\theta), r \sin(\theta)) r dr d\theta$$



Be able to do the same when the bounds of  $r$ , respectively  $\theta$ , depend on the other variable.  
 (100, 101)

35. (12.5) Applications of double integrals.

(a) Mass from density: If  $\rho(x, y)$  is the density (mass/area) of a laminar object with spatial extent given by a 2D region  $D$ , know the mass  $m$  is (102)

$$m = \iint_D \rho(x, y) dA$$

(b) Center of mass: Given the laminar object defined in (a), be able to compute the center of mass, or balance point,  $(\bar{x}, \bar{y})$  of the object from the formulas

$$\bar{x} = \frac{1}{m} \iint_D x \rho(x, y) dA \quad \text{and} \quad \bar{y} = \frac{1}{m} \iint_D y \rho(x, y) dA$$

(c) Probability: Let  $(X, Y)$  be a pair of random events that could be observed to take on values in a given region  $R$ , where the likelihood of the events is given by  $p(x, y)$  is a probability density function. Know the properties of the probability density: (a)

$$p(x, y) \geq 0 \quad \text{and} \quad \iint_R p(x, y) dA = 1$$

$$\iint_R p(x, y) dA = 1$$

(d) Let  $D$  be a subset of events of  $R$ . Know how to interpret a verbal/written description of  $D$  and formulate it in set notation:  $D = \{(x, y) | \dots\}$ .

(e) Be able to compute the probability  $P((x, y) \in D)$  of the events  $D$  as computed by the double integral: (b, 105)

$$P((x, y) \in D) = \iint_D p(x, y) dA$$

$z z_D$

$p(x, y) dA$ .

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## Example Questions

1. (6.4) Consider the curve defined parametrically by  $y = e^{-t}\cos(t)$ ,  $x = e^{-t}\sin(t)$  from  $t = 0$  to  $t = 2\pi$ .

Calculate the arc length.

Solution:

$$dx$$

$$dt = -e^{-t}\sin(t) + e^{-t}\cos(t) = e^{-t}(\cos(t) - \sin(t))$$

$$dy$$

$$dt = -e^{-t}(\cos(t) + \sin(t))$$

$$\sqrt{dx^2 + dy^2}$$

$$= e^{-2t}(\cos(t) - \sin(t))^2 + e^{-2t}(\cos(t) + \sin(t))^2$$

$$= e^{-2t}(\cos^2(t) - 2\cos(t)\sin(t) + \sin^2(t) + \cos^2(t) + 2\cos(t)\sin(t) + \sin^2(t))$$

$$= e^{-2t}(2\cos^2(t) + 2\sin^2(t))$$

$$= 2e^{-2t}$$

$$= 2e^{-2t}$$

$$= 2e^{-2t}$$

$$L = \int_0^{2\pi} 2e^{-2t} dt$$

$$= \int_0^{2\pi} 2e^{-2t} dt$$

$$= \int_0^{2\pi} 2e^{-2t} dt$$

$$= \int_0^{2\pi} 2e^{-2t} dt$$

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$$= \int_0^{2\pi} 2e^{-2t} dt$$

$$= \int_0^{2\pi} 2e^{-2t} dt$$

2. (6.4) Set-up (don't evaluate) the integrals for the arc length for the following curves.

(a)  $x = t^2$  and  $y = \cos(t)$  for  $0 \leq t \leq \pi$

Solution:

$$\sqrt{dx^2 + dy^2}$$

$$= \sqrt{2t dt^2 + (-\sin t dt)^2}$$

$$= \sqrt{2t^2 + \sin^2 t} dt$$

$$= \int_0^\pi \sqrt{2t^2 + \sin^2 t} dt$$

$$= \int_0^\pi \sqrt{2t^2 + \sin^2 t} dt$$

$$= \int_0^\pi \sqrt{2t^2 + \sin^2 t} dt$$

$$L = \int_0^1 \sqrt{4t^2 + \sin(t)^2} dt$$

(b)  $x(t) = \ln(t + 1) \cos(t)$  and  $y(t) = \ln(t + 1) \sin(t)$  for  $0 \leq t \leq 2\pi$

Solution: The derivatives:

$$x'(t) = -\ln(t + 1) \sin(t) + \cos(t)$$

$$y'(t) = \ln(t + 1) \cos(t) + \sin(t)$$

So,

$$L = \int_0^{2\pi} \sqrt{(-\ln(t + 1) \sin(t) + \cos(t))^2 + (\ln(t + 1) \cos(t) + \sin(t))^2} dt$$

3. (6.4) Set-up (don't evaluate) the integrals for the arc length for the following curves.

(a)  $x = y^2 - 1$  for  $-1 \leq y \leq 1$

Solution:

$$L = \int_{-1}^1 \sqrt{1 + 4y^2} dy$$

$$L = \int_{-1}^{1} \sqrt{1+y^2} dy$$

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(b)  $y = xe^x$  for  $0 \leq x \leq 1$

Solution:

$$L = \int_0^1 \sqrt{1 + (e^x + xe^x)^2} dx$$

4. (6.5) Find the average value of the following functions over the given interval.

(a)  $f(x) = 4 - x^2$  over  $[-2, 2]$

Solution:

$$f_{ave} = \frac{1}{2 - (-2)} \int_{-2}^2 (4 - x^2) dx = \frac{1}{4} \left[ 4x - \frac{x^3}{3} \right]_{-2}^2 = \frac{1}{4} \left( 8 - \frac{8}{3} - (-8 + \frac{8}{3}) \right) = \frac{1}{4} \left( 16 - \frac{16}{3} \right) = \frac{8}{3}$$

(b)  $f(x) = x \sin(x)$  over  $[0, 2\pi]$

Solution:

$$f_{ave} = \frac{1}{2\pi} \int_0^{2\pi} x \sin(x) dx = \frac{1}{2\pi} \left[ -x \cos(x) + \sin(x) \right]_0^{2\pi} = \frac{1}{2\pi} \left( -2\pi \cos(2\pi) + \sin(2\pi) - (0 \cos(0) + \sin(0)) \right) = \frac{1}{2\pi} (-2\pi) = -1$$

$$= \frac{1}{2\pi}(-2\pi + 0) = -1$$

using integration by parts.

5. (6.5) Consider the function  $f(x) = x^3 - x + 1$  on  $[-1, 1]$ .

(a) Find the average value.

Solution:

$$f_{ave} = \frac{1}{2} \int_{-1}^1 f(x) dx = 1$$

(b) Find all  $c$ , such that  $f(c) = f_{ave}$

Solution:

$$1 = f_{ave} = f(x) = c^3 - c + 1 \quad c^3 - c = 0 \quad c = -1, 0, 1$$

(c) Sketch the graph of  $f$  and a rectangle with a base  $[-1, 1]$ , whose area is the same as the area under graph of  $f$ .

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1

2

y

x

-2 -1 1 2 -1

6. (6.6) Compute the work required to lift a load 50kg up a 20-meter building using a cable. The cable weights 1kg/meter, and is spooled up as it is pulled up with the load.

Solution: The mass  $m$  of the system as a function of  $y = 0$  to  $y = 20$  is  $m(y) = 50+20-y = 70-y$ . The force is  $F = gm(y)$ . The work is

$$W = \int_0^{20} g(70 - y)dy = g70 \times 20 - \frac{g}{2}20^2.$$

7. A conical tank as the pointy end at  $y = 0$  and height  $y = 2$  meters. The top of the cone has an upper radius  $r = 1$ . The tank is filled with water. Set up the integral that calculates the work required to pump the water out over the top of the tank, but do not calculate. Water has mass density  $1000 \text{ kg/m}^3$ .

Solution: The radius of the tank as a function of  $y$  is  $r(y) = \frac{1}{2}y$ . This makes  $r(0) = 0$  and  $r(2) = 1$ , as required. The volume of water at  $y$  for a thin  $dy$  segment is  $dV = \pi r^2(y)dy$ , so the mass at each  $dy$  segment is  $dm = 1000\pi r^2(y)dy$ . The force of gravity is  $F = \text{mass} \times g$  and the work to move it from  $y$  to 2, is  $dW = g(2 - y)1000\pi r^2(y)dy$ . The total work is found by summing all  $dW$  work from  $y = 0$  to  $y = 2$ .

$$W = \int_0^2 g(2 - y)1000\pi \frac{1}{4}y^2 dy.$$

8. (6.6) Compute the work required to launch a 1200 kg satellite vertically into the orbit of 20,000 km. Assume the mass of the earth is  $M = 5.97 \times 10^{24} \text{ kg}$ , the radius of the earth is  $R = 6371 \text{ km}$  and the gravitational constant  $G = 6.67 \times 10^{-11} \frac{\text{N m}^2}{\text{kg}^2}$ .

Solution: The satellite has to be lifted from  $R = 6371 \text{ km}$  to 20,000 km and the gravitational force

is  $F(r) = G \frac{mM}{r^2}$ , where  $m = 1200 \text{ kg}$  is the mass of the satellite. So the work required is

$$\int_{20,000 \text{ km}}^{6371 \text{ km}} -GmM \frac{1}{r^2} dr$$

$$W = -GmM \int_{6371 \text{ km}}^{20,000 \text{ km}} \frac{1}{r^2} dr = \approx 5.11 \times 10^{-13} \text{ J}$$

9. Calculate the work to compress a (Hookean) spring that exerts a force  $F(x) = -2x$  when the spring's position  $x$  is compressed away from it's starting neutral point  $x = 0$  up to  $x = 3$ .

Solution:

$$W = - \int_0^3 2x dx = [-x^2]_0^3 = -9.$$

A positive work is done by the compressor, the spring does negative work. Either sign will be accepted.

10. Calculate the total force on one of the sidewalls of a two-meter cube tank of water. Note, the pressure  $P$  at every depth  $d$  is  $P = \rho g d$ , where  $\rho = 1000 \text{ kg/m}^3$ . Set up the integral for the force, but do not calculate.

Solution: The length of a single sidewall is  $\ell = 2$  meters, so the force on each  $dy$  segment at depth  $d = 2 - y$  is  $dF = \rho g(2 - y)2dy$ . Recall that  $2dy$  is the area, and the pressure at that depth is  $\rho g(2 - y)$ . The total force is the integral of the force at each depth from  $y = 0$  to  $y = 2$ :

$$F_{total} = \int_0^2 dF = \int_0^2 \rho g(2 - y)2dy.$$

11. Write out the describe sequence  $a_n$  indexed by  $n$  in set notation:  $\{??\}_{n=?}^?$ . Indicate start and ending indexes (if there is no end, indicate  $\infty$ ).

- (a)  $a_n$  counts up by one, starting from 1 to infinity.
- (b)  $a_n$  counts up by two, starting at 6 to infinity.
- (c)  $a_n$  is all positive odd numbers starting at one to infinity.
- (d)  $a_n$  has a positive one at the start  $n = 1$ , and alternates sign for all  $n$  increasing.



Solution:

$$(a) \{n\}_{n=1}^{\infty}$$

$$(b) \{2n\}_{n=3}^{\infty}$$

$$(c) \{2n - 1\}_{n=1}^{\infty}$$

$$(d) \{(-1)^{n+1}\}_{n=1}^{\infty}$$

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12. Match the equivalent sequences for  $n = 0, 1, 2, 3, 4, 5, 6 \dots$ , or state that there is no match. Note that  $(-1)^{12} = i$ .

(a)  $a_n = \sin(n\pi/2)$

(b)  $b_n = \cos(n\pi/2)$

(c)  $c_n = (-1)^{n/2}, n = 0,$

$\text{evens } 0, \text{ else}$

(d)  $d_n = (-1)^{(2n+1)/2}, n$

$= \text{odds } 0, n = \text{odds}$

$( \text{ else}$

(e)  $e_n = 0, \text{ else}$

Solution:  $b_n = c_n, a_n = e_n,$  and  $d_n$  is unmatched.

13. (8.2) The given series are telescoping series. Find the partial sums and decide whether the series is convergent or divergent. If it is convergent, find its sum.

$$\sum_{k=1}^{\infty} \frac{1}{k(k+2)}$$

Solution: Write the summand as a difference

$$\frac{k(k+2)}{2} = \frac{A}{k+B}$$

$$k + 2^2 = Ak + 2A + Bk \quad A = 1 \quad B = -1$$

So the  $n$ th partial sum is

$$\begin{aligned}
 s_n &= \sum_{k=1}^n \frac{k(k+2)}{2} \\
 &= \sum_{k=1}^n \left( \frac{k^2}{2} + k \right) \\
 &= \frac{1}{2} \sum_{k=1}^n k^2 + \sum_{k=1}^n k \\
 &= \frac{1}{2} \left( \frac{n(n+1)(2n+1)}{6} \right) + \frac{n(n+1)}{2} \\
 &= \frac{n(n+1)(2n+1)}{12} + \frac{6n(n+1)}{12} \\
 &= \frac{n(n+1)(2n+1 + 6)}{12} \\
 &= \frac{n(n+1)(2n+7)}{12}
 \end{aligned}$$

Now the limit of the partial sums exists and it is

$$\begin{aligned}
 X^\infty &= \frac{1}{12} \lim_{n \rightarrow \infty} n(n+1)(2n+7) \\
 &= \frac{1}{12} \lim_{n \rightarrow \infty} (2n^3 + 9n^2 + 7n) \\
 &= \infty
 \end{aligned}$$

$$k(k+2) = \lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} \sum_{k=1}^{n+2} \frac{1}{2^k} = 1 + \frac{1}{2} = \frac{3}{2}$$

Solution: Write each log ratio term as a difference of logs:

$$\ln \frac{n}{n+1} = \ln(n) - \ln(n+1)$$

So, the partial sum to  $N$  can be expressed as

$$s_N = \sum_{n=2}^N \ln \frac{n}{n+1} = [\ln(2) - \ln(3)] + [\ln(3) - \ln(4)] + \dots + [\ln(N) - \ln(N+1)]$$

$$= \ln(2) + [-\ln(3) + \ln(3)] + [-\ln(4) + \ln(4)] + \dots - \ln(N+1)$$

$$= \ln(2) - \ln(N+1)$$

$\lim_{N \rightarrow \infty} s_N = \lim_{N \rightarrow \infty} [\ln(2) - \ln(N+1)] = -\infty,$

and so the series is divergent.

$$(c) \sum_{k=2}^{\infty} \ln \frac{k^2 - 1}{k^2 + 2k}$$

Solution: Write the summand as a difference

$$\ln \frac{k^2 - 1}{k^2 + 2k} = \ln(k^2 - 1) - \ln(k^2 + 2k) = \ln(k^2 - 1) - \ln((k + 1)^2 - 1)$$

So the  $n$ th partial sum for  $n \geq 2$  is

$$s_n = \sum_{k=2}^n \ln \frac{k^2 - 1}{k^2 + 2k} = \sum_{k=2}^n \ln(k^2 - 1) - \sum_{k=2}^n \ln((k + 1)^2 - 1)$$

$$= \ln(2) - \ln(3) + \ln(3) - \ln(4) + \dots + \ln(n) - \ln((n + 1)^2 - 1)$$

$$= \ln(2) - \ln((n + 1)^2 - 1)$$

Now

$$\lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} \ln(2) - \ln((n + 1)^2 - 1) = -\infty$$

and so the series is divergent.

14. (8.2) Decide whether the given geometric series is convergent or divergent. If it is convergent, find its sum.

(a)  $\sum_{k=0}^{\infty} 3^{2k+1} 2^{-k}$

Solution:  $\sum_{k=0}^{\infty}$

$$3^{2k+1} 2^{-k} = 3 \cdot 9^k \cdot 2^{-k} = 3 \left(\frac{9}{2}\right)^k$$

So the common ratio  $r = \frac{9}{2} > 1$  and the series is divergent.

$$(b) \sum_{k=2}^{\infty} \frac{3^{3k-1}}{(-2)^k}$$

Solution:  $\sum_{k=2}^{\infty} \frac{3^{3k-1}}{(-2)^k}$

$$\sum_{k=2}^{\infty} \frac{3^{3k-1}}{(-2)^k} = \sum_{k=2}^{\infty} \frac{3^{3k} \cdot 3^{-1}}{(-2)^k} = \sum_{k=2}^{\infty} \frac{3^k \cdot 27}{(-2)^k}$$

$$= \sum_{k=2}^{\infty} \frac{3^k \cdot 27}{(-2)^k} = \sum_{k=2}^{\infty} \left(\frac{3 \cdot 27}{-2}\right)^k$$

$$= \sum_{k=2}^{\infty} \left(\frac{81}{-2}\right)^k$$

$$= \sum_{k=2}^{\infty} \left(-\frac{81}{2}\right)^k$$

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since the common ratio is  $r = -\frac{81}{2}$ ,  $|r| > 1$  and the series is divergent.

15. (8.2) Decide whether the following series are divergent or convergent.

$$(a) \sum_{k=1}^{\infty} \frac{1}{k+1} 2^k$$

Solution:  $\sum_{k=1}^{\infty} \frac{1}{k}$  is divergent, since it is a harmonic series, and  $\sum_{k=1}^{\infty} \frac{1}{2^k}$  is convergent, since it is a geometric series with common ratio  $\frac{1}{2}$ . So  $\sum_{k=1}^{\infty} \frac{1}{k+2^k}$  has to be divergent.

(b)  $\sum_{k=1}^{\infty} \frac{2}{k+2^k}$

Solution:  $\sum_{k=1}^{\infty} \frac{2}{k}$  is convergent, since it is a  $p$ -series with  $p > 1$ , and  $\sum_{k=1}^{\infty} \frac{1}{2^k}$  is convergent, since it is a geometric series with common ratio  $\frac{1}{2}$ . So  $\sum_{k=1}^{\infty} \frac{2}{k+2^k} = \sum_{k=1}^{\infty} \frac{2}{k} + \sum_{k=1}^{\infty} \frac{1}{2^k}$  is convergent.

16. (8.2) Use the Test for Divergence on the following series. Decide whether it shows the series is divergent or it is inconclusive.

$$\left( \sum_{k=1}^{\infty} \frac{1}{k} \right)$$

$$k^2 + 1$$

Solution:

$$\lim_{k \rightarrow \infty} \frac{k}{k^2 + 1} = \lim_{k \rightarrow \infty} \frac{1}{k + \frac{1}{k}} = 0$$

So the test is inconclusive.

$$\sum_{k=1}^{\infty} k \cos(k)$$

Solution: The limit  $\lim_{k \rightarrow \infty} k \cos(k)$  does not exist, so the series is divergent.

$$(c) \sum_{k=1}^{\infty} \frac{1}{\cos k}$$

Solution:

$$\lim_{k \rightarrow \infty} \frac{1}{\cos k} = 1$$

So the series is divergent.

17. (8.3) Use the integral test to decide whether the following series are divergent or convergent. Make sure to check, that it is possible to use the integral test.

$$\sum_{k=1}^{\infty} \frac{1}{k^2 + 1}$$

Solution: The function  $\frac{x}{x^2+1}$

is positive and continuous. It is decreasing, since

$$(x^2 + 1)^2 = 1 - x^2$$

$$\frac{d}{dx} \left( \frac{x}{x^2 + 1} \right) = \frac{1(x^2 + 1) - x(2x)}{(x^2 + 1)^2}$$

d

x

$$(x^2 + 1)^2 < 0 \text{ for } x > 1$$

dx

$Z_\infty$

$Z_\infty$

x

1

$$\int_1^\infty \frac{1}{2 \ln |u|} du = \infty$$

1

$$\int_0^1 \frac{x}{x^2 + 1} dx =$$

0

u

1

using substitution  $u = x^2 + 1$ ,  $du = 2x dx$ . Since the integral diverges, the series is divergent.

(b)  $\sum_{k=2}^{\infty} \frac{k(\ln(k))^2}{1}$

1

Solution: The function  $\frac{1}{x \ln(x)^2}$

is positive and continuous. It is decreasing, since the denominator is increasing.

$$\int_1^\infty \frac{1}{u} du = \infty$$

$Z_\infty$

$Z_\infty$

1

=1

$$\int_1^\infty \frac{1}{x \ln(x)^2} dx =$$

1

$$\int_{\ln(2)}^{\ln(2)} \frac{1}{u^2} du =$$

2

$\ln(2)$

$\ln(2)$

$\ln(2)$



using substitution  $u = \ln(x)$ ,  $du = \frac{1}{x} dx$ . Since the integral converges, the series is convergent.

$$(c) \sum_{k=1}^{\infty} k^2$$

Solution: It is not possible to use the integral test, since the sequence  $\{k^2\}_{k=1}^{\infty}$  is increasing and thus any function, that takes these values at the integers, can't be decreasing.

18. (8.3) Use the comparison test to compare the given series to the indicated series. Decide

- (i) whether the indicated series is convergent or divergent
- (ii) whether the comparison test shows the given series is convergent, divergent or if the comparison test is inconclusive

(a)  $\sum_{k=1}^{\infty} \frac{k+1}{k}$  to  $\sum_{k=1}^{\infty} \frac{1}{k}$

Compare  $\sum_{k=1}^{\infty} \frac{1}{k}$

Solution:

(i)  $\sum_{k=1}^{\infty} \frac{1}{k}$  is the harmonic series, so it is divergent.

(ii) Now  $\frac{1}{k+1} < \frac{1}{k}$

$$k+1 > k \Rightarrow \frac{1}{k+1} < \frac{1}{k}$$

So the comparison test is inconclusive.

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(b)  $\sum_{k=0}^{\infty} (5^k + 3) \cdot 3^k$  to  $\sum_{k=0}^{\infty} 3 \cdot 5^k$

Compare  $\sum_{k=0}^{\infty} 3 \cdot 5^k$

Solution:

(i)  $\sum_{k=0}^{\infty} 3^k$  is a geometric series with common ratio  $r = \frac{3}{5} < 1$ , so the series is convergent.

(ii) Now

$$5^k + 3 \leq 3^k$$

$$5^k =$$

So by the comparison test,  $\sum_{k=0}^{\infty} 3^k$  is convergent.

(c) Compare  $\sum_{k=0}^{\infty} 2^k - 4$  to  $\sum_{k=0}^{\infty} 3^{k+1}$

Solution:

(i)  $\sum_{k=0}^{\infty} 3^k$  is a geometric series with common ratio  $r = \frac{3}{2} > 1$ , so the series is divergent.

(ii) Now

$$2^k - 4 \geq 3^k$$

$$2^k \geq 3^k$$

So by the comparison test,  $\sum_{k=0}^{\infty} \frac{3^{k+1}}{2^k - 4}$  is divergent.

19. (8.3) Use the limit comparison test to decide whether the given series are convergent or divergent. Use either a geometric series or a  $p$ -series to compare the series to.

(a)  $\sum_{k=1}^{\infty} \frac{k^3 - \sqrt[4]{2k}}{\sqrt{k^2 + 1}}$

Solution: To find a series to compare this series to, take the highest powers of numerator

and denominator, that is  $\sum_{k=1}^{\infty} \frac{k^3}{k^2}$

$$\sum_{k=1}^{\infty} \frac{k^3}{k^2}$$

$\sum_{k=1}^{\infty} \frac{1}{k^2}$ . This is a  $p$ -series with  $p = 2 > 1$ , so it is convergent.

Now using the limit comparison test

$$\lim_{k \rightarrow \infty} \frac{\frac{k^3 - \sqrt[4]{2k}}{\sqrt{k^2 + 1}}}{\frac{k^3}{k^2}} = \lim_{k \rightarrow \infty} \frac{k^3 - \sqrt[4]{2k}}{k^2 \sqrt{k^2 + 1}} = \lim_{k \rightarrow \infty} \frac{k^3 - \sqrt[4]{2k}}{k^2 (1 + \frac{1}{k^2})} = \lim_{k \rightarrow \infty} \frac{k^3 - \sqrt[4]{2k}}{k^2 - 2^{\frac{1}{4}} k^{\frac{1}{4}}} = 1$$

$$\sum_{k=1}^{\infty} \frac{k^3}{\sqrt{k^2+1}}$$

So  $\sum_{k=1}^{\infty} k^3 - \sqrt{2k^k}$  is also convergent.

(b)  $\sum_{k=0}^{\infty} \frac{3}{2^k - k}$

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Solution: Since  $2^k$  grows faster than  $k$ , compare this series to  $\sum_{k=0}^{\infty} \frac{1}{2^k}$ . This is a geometric series with  $r = \frac{1}{2} < 1$ , so it is convergent. Now using the limit comparison test

$$\lim_{k \rightarrow \infty} \frac{\frac{3}{2^k - k}}{\frac{3}{2^k}} = \lim_{k \rightarrow \infty} \frac{2^k}{2^k - k} = \lim_{k \rightarrow \infty} \frac{1}{1 - \frac{k}{2^k}} = 3$$

since  $\lim_{k \rightarrow \infty} \frac{k}{2^k} = 0$ . So  $\sum_{k=0}^{\infty} \frac{3}{2^k - k}$  is convergent.

20. (8.4) Use the alternating series test on the following series. Decide whether it shows the series is

convergent or if the test is inconclusive.

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2 + 1}$$

Solution: To see the sequence  $b_k = \frac{1}{k^2 + 1}$

$\frac{1}{k^2 + 1}$  is decreasing, consider the function  $f(x) = \frac{1}{x^2 + 1}$ . If

this function is decreasing, then the sequence is decreasing. Now

$$f'(x) = \frac{0 \cdot (x^2 + 1) - x(2x)}{(x^2 + 1)^2} = \frac{-2x^2}{(x^2 + 1)^2} < 0 \text{ for } x > 1$$

So the function and especially the sequence is decreasing. Also

$$\lim_{k \rightarrow \infty} b_k = \lim_{k \rightarrow \infty} \frac{1}{k^2 + 1} = 0$$

So by the alternating series test the given series is convergent.

(b)  $\sum_{k=1}^{\infty} \frac{(-1)^k (k+2)}{k}$

Solution: Consider the sequence  $b_k = \frac{k+2}{k}$

$\frac{k+2}{k}$ . It is

$\frac{k+2}{k}$

$\frac{k+2}{k} = 1 + \frac{2}{k}$

$\lim_{k \rightarrow \infty} \frac{k+2}{k} = 1$

So this series does not satisfy the conditions of the alternating series test and the test is inconclusive.

21. (8.4) Use the ratio test on the following series. Decide whether it shows the series is convergent, divergent or if the test is inconclusive.

$$(a) \sum_{k=1}^{\infty} k! 3^k$$

Solution:

$$\lim_{k \rightarrow \infty} \frac{(k+1)!}{k!} \frac{3^{k+1}}{3^k} = \lim_{k \rightarrow \infty} (k+1) 3 = \infty$$

So the series is divergent.

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$$\sum_{k=1}^{\infty} \frac{k^2}{k^4 - \sqrt{k}}$$

Solution:

$$\lim_{k \rightarrow \infty} \frac{(k+1)^2}{k^2} \frac{(k+1)^4 - \sqrt{k+1}}{k^4 - \sqrt{k}} = 1$$

$$\lim_{k \rightarrow \infty} \frac{k^2}{k^4 - \sqrt{k}}$$

since in each fraction the highest power in numerator and denominator are the same. So the root test is inconclusive.

$$\lim_{k \rightarrow \infty} \frac{1}{k} = 0$$

Solution:

$$\lim_{k \rightarrow \infty} \frac{(-1)^{k+1}(k+1)}{2^{k+1} + (k+1)^2} = \lim_{k \rightarrow \infty} \frac{k+1}{2^{k+1} + (k+1)^2}$$

$$= \lim_{k \rightarrow \infty} \frac{1}{2 + \frac{(k+1)^2}{2^{k+1}}}$$

$$= \lim_{k \rightarrow \infty} \frac{1}{2 + \frac{(k+1)^2}{2^{k+1}}}$$

$$= \frac{1}{2}$$

Since  $\lim_{k \rightarrow \infty} \frac{(k+1)^2}{2^{k+1}} = 0$

$$2^k = 0 = \lim_{k \rightarrow \infty} 2^{-k}. \text{ So the series is convergent.}$$

22. Consider the converging p-series  $s = \sum_{n=1}^{\infty} \frac{1}{n^2}$ .

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

(a) Find the remainder of the  $s_9$  partial sum. Express using summation notation.

Solution:

$$R_9 = s - s_9 = \sum_{n=10}^{\infty} \frac{1}{n^2}$$

(b) Using the integral remainder estimate, estimate how close  $s_9$  is to  $s$ .

Solution:

$$\int_9^{\infty} \frac{1}{x^2} dx = 1/9,$$

$$\int_{10}^{\infty} \frac{1}{x^2} dx = 1/10.$$

So  $1/10 \leq |R_9| \leq 1/9$

(c) Is  $s_9$  within 0.1 of  $s$ ?

Solution: no,  $s_9$  is more than a tenth from  $s$  because the error/remainder is larger than  $1/10$ .

23. Consider the converging alternating series  $s = \sum_{n=0}^{\infty} (-1)^n e^{-\sqrt{n}}$ .

$$\sum_{n=0}^{\infty} (-1)^n e^{-\sqrt{n}}$$

(a) Find the remainder of the  $s_{15}$  partial sum. Express using summation notation.



Solution:

$$R_{15} = s - s_{15} = \sum_{n=16}^{\infty} (-1)^n e^{-\sqrt{n}}$$

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(b) Using the alternating series remainder estimate, estimate how close  $s_{15}$  is to  $s$ .

Solution:

$$|R_{15}| = |a_{16}| = e^{-\sqrt{16}} = e^{-4} \approx 0.0183.$$

(c) Is  $s_{15}$  within 1/16th of  $s$ ?

Solution: Yes,  $1/16 = 2^{-4} = 0.0625 > e^{-4} = 0.0183$ .

24. (8.5) Find the interval of convergence and the radius of convergence for the following power series.

$$\sum_{k=0}^{\infty} \frac{x^k}{k!}$$

2. This converges for  $\frac{x+3}{2} < 1$

Solution: This is a geometric series with common ratio  $r = \frac{x+3}{2}$

$\geq 1$ . So it converges if and only if  $|x + 3| < 2$ , that is

and diverges for  $\frac{x+3}{2} \geq 1$

2

2

$$-2 < x + 3 < 2 \implies -5 < x < -1$$

So the radius of convergence is 2 and the interval of convergence is  $(-5, -1)$ .

(b)  $\sum_{k=1}^{\infty} (2x +$

$$1)^k \quad 3^{2k-1} k$$

Solution: Use the ratio test

$$\lim_{k \rightarrow \infty} \frac{(2x+1)^{k+1}}{(2x+1)^k} = \lim_{k \rightarrow \infty} \frac{3^{2(k+1)-1} (k+1)}{3^{2k-1} k} = |2x+1| \frac{k+1}{k}$$

So the test gives, that the series converges for  $|2x + 1| < 9$ , diverges for  $|2x + 1| > 9$  and it is inconclusive for  $|2x + 1| = 9$ . So now test  $x = 4$  and  $x = -5$  separately.

$$x = 4 : \sum_{k=1}^{\infty} \frac{3^{2k-1}}{9^k} = \sum_{k=1}^{\infty} \frac{3}{9^k}$$

This series is a multiple of the harmonic series, so it diverges.

$$x = -5 : \sum_{k=1}^{\infty} \frac{3^{2x-1} (-9)^k}{k} = \sum_{k=1}^{\infty} \frac{3(-1)^k}{k}$$

This series is a multiple of the alternating harmonic series, so it converges. So the interval of convergence for the power series is  $[-5, 4)$  and the radius of convergence is  $\frac{9}{2}$ .

$$(c) \sum_{k=0}^{\infty} 2)^k k!$$

Solution: Use the ratio test

$$\lim_{k \rightarrow \infty} \frac{(k+1)(x-2)^{k+1}}{k!} = \lim_{k \rightarrow \infty} \frac{(k+1)!}{k!} |x-2| = \lim_{k \rightarrow \infty} \frac{(k+1)(x-2)^{k+1}}{k(x-2)^k} = \lim_{k \rightarrow \infty} \frac{(k+1)|x-2|}{k} = 0$$

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So the series converges for all  $x$ . That is the interval of convergence is  $(-\infty, \infty)$  and the radius of convergence is  $R = \infty$ .

25. (8.6) Use  $1 - x = \sum_{k=0}^{\infty} x^k$  to find a power series for the following functions.  $k=0$
- (a)  $3 - x$

Solution:

$$3 - x = 1 + 2 + \sum_{k=2}^{\infty} x^k = 1 + 2 + \sum_{k=2}^{\infty} \frac{1}{3} 3^k x^k = 1 + 2 + \frac{1}{3} \sum_{k=2}^{\infty} (3x)^k = 1 + 2 + \frac{1}{3} \left( \frac{(3x)^2}{1 - 3x} \right) = 1 + 2 + \frac{1}{3} \frac{9x^2}{1 - 3x} = 1 + 2 + \frac{3x^2}{1 - 3x} = \frac{1 - 3x + 3x^2}{1 - 3x}$$

$$\sum_{k=0}^3 x^k$$

(b)  $\frac{x^2}{2+x}$

Solution:

$$\frac{x^2}{2+x} = x^2 \cdot \frac{1}{2+x} = x^2 \cdot \frac{1}{2} \cdot \frac{1}{1 + \frac{x}{2}} = \frac{x^2}{2} \sum_{k=0}^{\infty} (-1)^k \left(\frac{x}{2}\right)^k = \sum_{k=2}^{\infty} \frac{(-1)^{k-2} x^k}{2^{k-1}}$$

(c)  $\frac{1+x}{1-x}$

Solution:

$$\frac{1+x}{1-x} = (1+x) \sum_{k=0}^{\infty} x^k = \sum_{k=0}^{\infty} x^k + \sum_{k=0}^{\infty} x^{k+1} = \sum_{k=0}^{\infty} x^k + \sum_{k=1}^{\infty} x^k = 1 + \sum_{k=1}^{\infty} 2x^k = 1 + 2 \sum_{k=1}^{\infty} x^k$$

Note that  $\sum_{k=0}^{\infty} x^{k+1} = \sum_{k=1}^{\infty} x^k$ , and  $\sum_{k=0}^{\infty} x^k = 1 + \sum_{k=1}^{\infty} x^k$ , so

$$\frac{1+x}{1-x} = 1 + 2 \sum_{k=1}^{\infty} x^k$$

26. (8.6) Use  $\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$  to find a power series in  $(x-a)$  for the following functions.  $k=0$

$2 + x$  with  $a = -1$

Solution:

$$1 - (-(x + 1)) = X^\infty$$

$$(-x + 1)^k = X^\infty$$

$$\frac{2 + x = 1}{1}$$

$$\sum_{k=0}^{\infty} (-1)^k (x + 1)^k$$

(b) <sup>1</sup>

$1 + x$  with  $a = 1$

Solution:

$$1 - (x - 1) = X^\infty$$

$$= \frac{1}{2} X^\infty$$

$$\frac{1 + x = 1}{1}$$

$$2 - (-(x - 1)) = \frac{1}{2} X^\infty$$

$$1 - (x - 1)$$

$$\sum_{k=0}^{\infty} 2^{k+1} (x - 1)^k$$

(c) <sup>4</sup>

$x^2 + 4x$  with  $a = -2$

Solution:

$$(x + 2)^2 = X^\infty$$

$$= -X^\infty$$

$$= X^\infty$$

$$x^2 + 4x = 4$$

$$(x + 2)^2 - 4 = -1$$

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{4^{k+1}} (x + 2)^{2k}$$

27. (8.6) Consider the power series  $\sum_{k=0}^{\infty} x^k$  with radius of convergence 1. Use differentiation and integration to find power series for the following functions. What is the radius of convergence? (a)  $(1 - x)^3$

Solution: It is  $\frac{d^3}{dx^3} \frac{1}{1-x}$

$$\frac{d}{dx} \frac{1}{1-x} = \frac{1}{(1-x)^2}$$

$$\frac{d^2}{dx^2} \frac{1}{1-x} = \frac{2}{(1-x)^3}$$

$$\frac{d^3}{dx^3} \frac{1}{1-x} = \frac{6}{(1-x)^4}$$

This gives

$$\sum_{k=0}^{\infty} \frac{6}{(k+2)(k+1)} x^k$$

Differentiated series have the same radius of convergence as the original series, so the radius of convergence is  $R = 1$ .

(b)  $\frac{1}{(1+x)^2}$

Solution: It is  $\frac{d}{dx} \frac{1}{1+x} = -\frac{1}{(1+x)^2}$

This gives  $\frac{1}{(1+x)^2} = -\frac{d}{dx} \frac{1}{1+x}$

$$= -\sum_{k=0}^{\infty} (-1)^k k x^{k-1} = \sum_{k=0}^{\infty} (-1)^k (k-1) x^k$$

The radius of convergence is  $R = 1$ .

(c)  $\ln(1+x^2)$

Solution: It is  $\int \frac{2x}{1+x^2} dx = \ln(1+x^2) + C$

Since  $\ln(1+0) = 0$ ,  $C = 0$  and it is  $\ln(1+x^2) = \int_0^x \frac{2t}{1+t^2} dt = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+2}}{k+1}$

$$\ln(1 + x^2) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+2}}{k+1} = \sum_{k=1}^{\infty} \frac{(-1)^{k-1} x^{2k}}{k}$$

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The radius of convergence is  $R = 1$ .

28. (8.7) Find the Taylor series of the given functions about the indicated points. Use Sigma notation to express it, if possible.

(a)  $f(x) = e^{2x}$  about  $x = 0$

Solution:

$$f^{(k)}(x) = 2^k e^{2x} \implies f^{(k)}(0) = 2^k e^{2 \cdot 0} = 2^k$$

This gives  $f^{(k)}(0) = 2^k$  and the Taylor series is

$$T(x) = \sum_{k=0}^{\infty} \frac{2^k x^k}{k!}$$

(b)  $f(x) = \sin(x)$  about  $x = \frac{\pi}{4}$  (Hint: Write the Taylor series using two or more sums.)



Solution:

$$f^0(x) = \cos(x) \quad f^1(x) = -\sin(x) \quad f^2(x) = -\cos(x) \quad f^3(x) = \sin(x) \quad f^4(x) = \cos(x)$$

This gives

$$f^{(k)}\left(\frac{\pi}{4}\right) = \begin{cases} \frac{\sqrt{2}}{2} & 2k = 4l \\ -\frac{\sqrt{2}}{2} & 2k = 4l + 1 \\ \frac{\sqrt{2}}{2} & 2k = 4l + 2 \\ -\frac{\sqrt{2}}{2} & 2k = 4l + 3 \end{cases}$$

and the Taylor series is

$$T(x) = \sum_{l=0}^{\infty} \frac{f^{(4l)}\left(\frac{\pi}{4}\right)}{(4l)!} x^{4l} + \sum_{l=0}^{\infty} \frac{f^{(4l+1)}\left(\frac{\pi}{4}\right)}{(4l+1)!} x^{4l+1} + \sum_{l=0}^{\infty} \frac{f^{(4l+2)}\left(\frac{\pi}{4}\right)}{(4l+2)!} x^{4l+2} + \sum_{l=0}^{\infty} \frac{f^{(4l+3)}\left(\frac{\pi}{4}\right)}{(4l+3)!} x^{4l+3}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} + \sum_{k=0}^{\infty} \frac{(-1)^{k+1} x^{2k+1}}{(2k+1)!}$$

(c)  $f(x) = x^5 + 1$  about  $x = -1$

Solution:

$$f'(x) = 5x^4 \quad f''(x) = 20x^3 \quad f'''(x) = 60x^2 \dots$$

This gives

$$\begin{aligned} T(x) &= 0 \cdot (x+1)^0 + \frac{5}{1}(x+1)^1 + \frac{-20}{2}(x+1)^2 + \frac{60}{6}(x+1)^3 \\ &\quad + \frac{-120}{24}(x+1)^4 + \frac{120}{120}(x+1)^5 \\ &= 5(x+1) - 10(x+1)^2 + 10(x+1)^3 - 5(x+1)^4 + (x+1)^5 \end{aligned}$$

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29. (8.7) Remember the hyperbolic functions

$$\sinh(x) = e^x - e^{-x}$$

$$2\cosh(x) = \frac{e^x + e^{-x}}{2}$$

(a) Find the first three derivatives of  $\sinh(x)$ , that is  $\frac{d}{dx}\sinh(x)$ ,  $\frac{d^2}{dx^2}\sinh(x)$  and  $\frac{d^3}{dx^3}\sinh(x)$ .

$$\frac{d^2}{dx^2}\sinh(x) \text{ and } \frac{d^3}{dx^3}\sinh(x).$$

Solution:

$$\frac{d}{dx}\sinh(x) = \cosh(x)$$

$$\frac{d^2}{dx^2}\sinh(x) = \sinh(x)$$

$\frac{d^3}{dx^3}\sinh(x) = \cosh(x)$

$$\frac{d^3}{dx^3}\sinh(x) = \cosh(x)$$

(b) What is the Maclaurin series of  $\sinh(x)$ ? Use the sum notation to write it down.

Solution: It is  $\sinh(0) = 0$  and  $\cosh(0) = 1$ , so the Maclaurin series of  $\sinh(x)$  is

$$0 + 1x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \dots = X^\infty$$

$$\sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}$$

(c) What is the radius of convergence of the Maclaurin series?

Solution: Use the ratio test

$$\lim_{k \rightarrow \infty} \frac{x^{2(k+1)+1}}{(2(k+1)+1)!} = \lim_{k \rightarrow \infty} \frac{x^{2k+1}}{(2k+1)!} = \lim_{k \rightarrow \infty} \frac{x^{2k+1}}{(2k+3)(2k+2)} = 0$$

So the radius of convergence is  $R = \infty$  and the Maclaurin series converges everywhere.

(d) Show, that the Maclaurin series is equal to  $\sinh(x)$ .

Solution: It is

$$\frac{d^n}{dx^n} \sinh(x) = \begin{cases} \sinh(x) & n \text{ even} \\ \cosh(x) & n \text{ odd} \end{cases}$$

Now it is for  $|x| \leq d$

$$|\sinh(x)| \leq e^x \leq e^d \quad |\cosh(x)| \leq e^x \leq e^d$$

and so

$$\frac{d^n}{dx^n} \sinh(x) \leq e^d$$

for all  $x$  with  $|x| \leq d$ . Set  $M = e^d$ . So Taylor's Inequality gives

$$|R_n(x)| \leq \frac{e^d}{(n+1)!} |x|^{n+1}$$

and

$$\lim_{n \rightarrow \infty} |R_n(x)| = e^d \lim_{n \rightarrow \infty} \frac{|x|^{n+1}}{(n+1)!} = 0$$

Since the limit of the remainder converges to 0, the Taylor polynomials converge to  $\sinh(x)$ . That is the Maclaurin series is equal to  $\sinh(x)$ .

30. (8.7) Use Taylor's inequality to show that the given function and their Taylor series are equal on the open interval of convergence. That is, show that  $|R_n(x)|$  goes to zero as  $n \rightarrow \infty$  by bounding  $|R_n(x)|$  by

an expression involving  $n$ .

(a)  $f(x) = \cos(x)$  with Taylor series  $\sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!}$ , which converges for all  $x$

Solution: The derivatives of  $\cos(x)$  are up to a sign either  $\sin(x)$  or  $\cos(x)$ . Both are bounded above by 1, so  $|f^{(n+1)}(x)| \leq 1$  for all  $x$ . So set  $M = 1$  and Taylor's inequality gives

$$|R_n(x)| \leq \frac{1}{(n+1)!} |x|^{n+1} \xrightarrow{n \rightarrow \infty} 0$$

So  $\cos(x)$  is equal to its Taylor series.

(b)  $f(x) = (x-1)^5$  with Taylor series  $x^5 - 5x^4 + 10x^3 - 10x^2 + 5x - 1$  which converges for all  $x$

Solution: The higher derivatives are zero, that is  $f^{(n+1)}(x) = 0$  for  $n \geq 5$ . So 0 is an upper bound and Taylor's inequality gives

$$|R_n(x)| \leq 0^{n \rightarrow \infty} \rightarrow 0$$

So  $(x-1)^5$  equals its Taylor series.

31. Consider  $f(x) = \sqrt{x}$ . Use the linear  $T_1(x)$  and quadratic  $T_2(x)$  truncated Taylor series, expanded about  $a = 1$  to approximate  $\sqrt{2}$ . Which approximation does better?

Solution: The zeroth, first, and second derivatives are

$$f(1) = 1, f'(1) = \frac{1}{2}, f''(1) = -\frac{1}{4}$$

So, the linear approximation is

$$T_1(x) = 1 + \frac{1}{2}(x-1),$$

and the quadratic is

$$T_2(x) = 1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2.$$

The approximations are

$$T_1(2) = 1 + 1/2 = 3/2 = 1.5 \approx \sqrt{2} = 1.4142 \dots,$$

and

$$T_2(2) = 1 + 1/2 - 1/8 = 11/8 = 1.3758 \approx \sqrt{2} = 1.4142 \dots$$

The errors:  $|T_1(2) - \sqrt{2}| = 0.0858$ , and  $|T_2(2) - \sqrt{2}| = 0.0392$ , so the quadratic approximation does better.

32. (8.7) Let  $f(x) = e^{-x}$  and  $a = 1$ . Use Taylor's inequality to find the smallest  $N$  so that the Taylor polynomial  $T_N(x)$  is within a 0.1 error bound of  $f(x)$  on the interval  $(0, 2)$ .

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Solution: The derivatives are  $f^{(n)}(x) = (-1)^n e^{-x}$ . For each  $n$ , the bound for the absolute value of the derivatives on the interval  $(0, 2)$  is found on its left edge:  $x = 0$ . That is,  $M = e^0 = 1 > |f^{(n)}(x)|$  for  $x \in (0, 2)$ . The Taylor inequality states

$$|f(x) - T_N(x)| \leq \frac{M}{(N+1)!} |x-1|^{N+1},$$

for  $|x-1| < 1$  (i.e.,  $x \in (0, 2)$ ). Therefore, the bound is

$$\frac{1}{(N+1)!}$$

The goal is to make the bound less than  $0.1 = 1/10$ , so  $(N+1)! > 10$ , the smallest  $N$  for which this is satisfied is  $N = 3$ :  $4! = 24 > 10$ .

33. (9.1) Decide whether the given equation describes a sphere. If yes, find its radius and center. (a)  $x^2 + y^2 + z^2 + 2z - 6x + 3 = 0$

Solution: It is

$$0 = x^2 + y^2 + z^2 + 2z - 6x + 3 = (x-3)^2 + y^2 + (z+1)^2 - 9 - 1 + 3 = 0$$

$$(x-3)^2 + y^2 + (z+1)^2 = 7$$

So the equation describes a sphere with radius  $\sqrt{7}$  and center  $(3, 0, -1)$ .

- (b)  $x^2 + 2y^2 = 4x - 2 + 4y - z^2$

Solution: It is

$$0 = x^2 + 2y^2 + z^2 - 4x - 4y + 2 = (x-2)^2 + 2(y-1)^2 + z^2 - 4 - 2 + 2$$

$$(x-2)^2 + 2(y-1)^2 + z^2 = 4$$

This does not describe a sphere, since  $(y-1)^2$  is weighted more than  $(x-2)^2$  and  $z^2$ . This describes an ellipsoid.

- (c)  $x^2 + y^2 + z^2 + 4z + 2y + 12 = 0$

Solution: It is

$$0 = x^2 + (y + 1)^2 + (z + 2)^2 - 1 - 4 + 12$$

$$x^2 + (y + 1)^2 + (z + 2)^2 = -7$$

This does not describe a sphere, since not tuple  $(x, y, z)$  satisfies this equation.

34. (9.1) Write the set of all the points, that lie in the described object.

(a) A cylinder about the  $x$ -axis with radius 2 and height from  $x = 0$  to  $x = 3$ .

Solution:

$$\{(x, y, z) | y^2 + z^2 = 4 \text{ and } 0 \leq x \leq 3\}$$

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(b) A sphere of radius 3 with center  $(0, 3, 1)$ .

Solution:

$$\{(x, y, z) | x^2 + (y - 3)^2 + (z - 1)^2 = 9\}$$

(c) A disk with center  $(3, 1, -5)$  of radius 2 parallel to the  $yz$ -plane.

Solution:

$$\{(x, y, z) | x = 3 \text{ and } (y - 1)^2 + (z + 5)^2 \leq 4\}$$

35. (9.2) Consider

$$\sim u = \langle 3, -2, 12 \rangle \quad \sim v = \langle -2, 5, 0 \rangle \quad \sim w = 3\tilde{i} - 3\tilde{k}$$

Evaluate the following expressions.

(a)  $\sim u + \sim v$

Solution:

$$\sim u + \sim v = \langle 1, 3, 12 \rangle$$

(b)  $3\sim w$

Solution:

$$3\sim w = 9\tilde{i} - 9\tilde{k}$$

(c)  $2\sim v - \sim w$

Solution:

$$2\vec{v} - \vec{w} = \langle -7, 10, 3 \rangle = -7\vec{i} + 10\vec{j} + 3\vec{k}$$

(d)  $|\vec{w}|$

Solution:

$$|\vec{w}| = \sqrt{9 + 0 + 9} = 3\sqrt{2}$$

36. (9.2) Consider  $\vec{u}$  and  $\vec{v}$  in the figure. Draw on the figure the following vector calculations (a)  $\vec{u} + \vec{v}$   
(b)  $\vec{u} - \vec{v}$   
(c)  $\vec{v} - \vec{u}$

$\vec{v}$

$\vec{u}$

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$\vec{u} + \vec{v}$

$\vec{v}$

$\vec{v} - \vec{u}$

$\vec{u}$

$\vec{u} - \vec{v}$

Solution:

37. (9.2) For the given vectors, find the unit vector with the same direction as the given vector. Express it in component form *and* using the standard basis.  
(a)  $\vec{u} = \langle 2, -3, 5 \rangle$

Solution:

$$|\tilde{u}| = \sqrt{4 + 9 + 25} = \sqrt{38}$$

So the unit vector is

$$\begin{aligned} |\tilde{u}| &= 1 \\ \tilde{u} &= \frac{1}{\sqrt{38}} \langle 2, -3, 5 \rangle = \frac{1}{\sqrt{38}} \langle 2, -3, 5 \rangle \\ &= \frac{1}{\sqrt{38}} (2\tilde{i} - 3\tilde{j} + 5\tilde{k}) \end{aligned}$$

(b)  $\tilde{v} = -\tilde{j} + 4\tilde{k}$

Solution:

$$|\tilde{v}| = \sqrt{1 + 16} = \sqrt{17}$$

So the unit vector is  $\tilde{v}$

$$\begin{aligned} \tilde{v} &= -\tilde{j} + 4\tilde{k} = \langle 0, -1, 4 \rangle \\ |\tilde{v}| &= 1 \\ \tilde{v} &= \frac{1}{\sqrt{17}} \langle 0, -1, 4 \rangle = \frac{1}{\sqrt{17}} (-\tilde{j} + 4\tilde{k}) \end{aligned}$$

(c)  $\tilde{w}$  the vector from  $P(0, 12, -3)$  to  $Q(3, 8, 1)$

Solution:

$$\tilde{w} = \langle 3, -4, 4 \rangle \quad |\tilde{w}| = \sqrt{9 + 16 + 16} = \sqrt{41}$$

So the unit vector is

$$\begin{aligned} |\tilde{w}| &= 1 \\ \tilde{w} &= \frac{1}{\sqrt{41}} \langle 3, -4, 4 \rangle = \frac{1}{\sqrt{41}} \langle 3, -4, 4 \rangle \end{aligned}$$



$$\sim w = \sqrt{41,4} \sqrt{41} >= 3 \sqrt{41} \tilde{i} - 4 \sqrt{41} \tilde{j} + 4 \sqrt{41} \tilde{k}$$

38. (9.3) Compute the dot product  $\sim u \cdot \sim v$  for the given vectors  $\sim u$  and  $\sim v$ .

(a)  $\sim u = \langle 3, 7, -1 \rangle$  and  $\sim v = \langle 2, 0, 3 \rangle$

Solution:

$$\sim u \cdot \sim v = 6 + 0 - 3 = 3$$

(b)  $\sim u = -2\tilde{i} + 4\tilde{k} + \tilde{j}$  and  $\sim v = \langle -1, 2, 1 \rangle$

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Solution:

$$\sim u \cdot \sim v = (-2)(-1) + 1 \cdot 2 + 4 \cdot 1 = 8$$

(c)  $\sim u$  the vector from  $P(1, 3, 2)$  to  $Q(2, 3, -2)$ ,  $\sim v = 5\tilde{i} - 3\tilde{k}$

Solution:

$$\sim u = \langle 1, 0, -4 \rangle \quad \sim u \cdot \sim v = 5 + 12 = 17$$

39. (9.3) Compute the angle between the given vectors  $\sim u$  and  $\sim v$ .

(a)  $\sim u = \langle 3, -2, 1 \rangle$  and  $\sim v = \langle 2, 0, -1 \rangle$

Solution:

$$\cos(\theta) = \frac{\sim u \cdot \sim v}{|\sim u||\sim v|} = \frac{6 - 1}{\sqrt{9 + 4 + 1} \sqrt{4 + 1}} = \frac{5}{\sqrt{14} \sqrt{5}}$$

$$\theta = \arccos \frac{5}{\sqrt{14} \sqrt{5}} \approx 0.93 = 53.3^\circ$$

(b)  $\tilde{u}$  the vector from  $P(3, -1, 4)$  to  $Q(-1, 3, 0)$  and  $\tilde{v}$  the vector from  $P$  to  $R(-2, 9, -1)$

Solution:

$$\tilde{PQ} = \langle -4, 4, -4 \rangle = 4 \langle -1, 1, -1 \rangle \quad \tilde{PR} = \langle -5, 10, -5 \rangle = 5 \langle -1, 2, -1 \rangle$$

Since the length does not have an effect on the angle, one can use multiples to calculate the angle

$$\cos(\theta) = \langle -1, 1, -1 \rangle \cdot \langle -1, 2, -1 \rangle$$

$$|\langle -1, 1, -1 \rangle| |\langle -1, 2, -1 \rangle| = \sqrt{1+2+1} \sqrt{1+4+1} = \sqrt{3} \sqrt{6} = 3\sqrt{2}$$

$$\theta = \arccos \frac{4}{3\sqrt{2}} \approx 0.34 = 19.47^\circ$$

40. (9.3) A sled is pulled along a horizontal path by a rope. A force of 30 N is applied to the rope at an angle of  $30^\circ$  from the horizontal. How much work is done, when the sled moved 20 m?

Solution: The force  $\tilde{F}$  has a magnitude of  $|\tilde{F}| = 30$  N and has an angle of  $30^\circ$  from the horizontal, so

$$\tilde{F} = 30 \text{ N} \cos(30^\circ) \tilde{i} + \sin(30^\circ) \tilde{j} = 30 \text{ N} \left( \frac{\sqrt{3}}{2} \tilde{i} + \frac{1}{2} \tilde{j} \right)$$

The sled moves along the horizontal, so the distance vector is  $\tilde{d} = 20 \text{ m} \tilde{i}$ . So the work is

$$W = \tilde{F} \cdot \tilde{d} = 30 \text{ N} \cdot 20 \text{ m} \cos(30^\circ) = 300 \sqrt{3} \text{ J} \approx 519.6 \text{ J}$$

41. (9.3) Consider vectors  $h\mathbf{3}, 2, -1\mathbf{i}$  and  $h\mathbf{1}, 2, c\mathbf{i}$ , where  $c$  is a constant. Find a value for  $c$  that makes the two vectors orthogonal.

Solution:  $h3, 2, -1i \cdot h1, 2, ci = 3 + 4 - c = 0 \Rightarrow c = 7$ .

42. (9.3) Consider the vectors  $\vec{u} = h3, 1i$ ,  $\vec{v} = h1, 2i$  and  $\vec{w} = h-1, 2i$ . Find numbers  $a$  and  $b$  in order represent the vector  $\vec{w}$  in terms of  $\vec{u}$  and  $\vec{v}$ :

$$\vec{w} = a\vec{u} + b\vec{v}.$$

Hint: take dot products of the above expression with  $\vec{u}$  and  $\vec{v}$  to solve for  $a$  and  $b$ . Verify your answer using direct computation.

Solution:

$$\begin{aligned} \vec{w} &= a\vec{u} + b\vec{v} & \vec{w} \cdot \vec{u} &= -1 \\ \vec{w} \cdot \vec{v} &= 3 & \vec{u} \cdot \vec{v} &= 5 \\ |\vec{u}|^2 &= 10 & |\vec{v}|^2 &= 5 \\ \vec{w} \cdot \vec{u} &= (a\vec{u} + b\vec{v}) \cdot \vec{u} = -1 = a10 + b5 \\ \vec{w} \cdot \vec{v} &= (a\vec{u} + b\vec{v}) \cdot \vec{v} = 3 = a5 + b5 \\ \Rightarrow -4 &= 5a \Rightarrow a = -4/5 \\ \Rightarrow 3 &= -4 + 5b \Rightarrow b = 7/5. \\ -4/5\vec{u} + 7/5\vec{v} &= -4/5h3, 1i + 7/5h1, 2i \\ h-12/5 + 7/5, -4/5 + 14/5i &= h-5/5, 10/5i = h-1, 2i = \vec{w} \end{aligned}$$

43. (9.3) Consider the vectors  $\vec{u}$  and  $\vec{v}$  in the figure. Draw on the figure the components  $\text{comp}_{\vec{u}}(\vec{v})$  and

$|\vec{v}|$

$|\vec{u}|$

$\text{comp}_{\vec{u}}(\vec{v})$

$\begin{array}{c} !v \\ \text{comp}_{!v}(!u) \\ !u \\ \text{comp}_{!u}(!v) \end{array}$

Solution:

44. (9.3) Consider the orthogonal vectors  $\sim{u} = h1, 1i$ ,  $\sim{v} = h-1, 1i$ . Verify by direct computation that the vector  $\sim{w} = h-1, 2i$  can be expressed as

$$\sim{w} = \text{proj}_{\sim{u}}(\sim{w}) + \text{proj}_{\sim{v}}(\sim{w}).$$

Note this representation of  $\sim{w}$  is only possible when  $\sim{u}$  and  $\sim{v}$ .

Solution:

$$\begin{aligned} \sim{w} \cdot \sim{u} &= 1 \\ \sim{w} \cdot \sim{v} &= 3 \\ |\sim{u}|^2 &= 2 \\ |\sim{v}|^2 &= 2 \\ \text{proj}_{\sim{u}}(\sim{w}) &= \frac{1}{2}h1, 1i \\ \text{proj}_{\sim{v}}(\sim{w}) &= \frac{3}{2}h-1, 1i \\ \text{proj}_{\sim{u}}(\sim{w}) + \text{proj}_{\sim{v}}(\sim{w}) &= h-1, 2i = \sim{w}. \end{aligned}$$

45. (9.4) Calculate the cross product of the given vector  $\sim{u}$  and  $\sim{v}$ .

(a)  $\sim{u} = \langle 3, -1, 4 \rangle$  and  $\sim{v} = \langle 2, -1, 6 \rangle$

Solution:

$$\langle 3, -1, 4 \rangle \times \langle 2, -1, 6 \rangle = \langle -2, -10, -1 \rangle$$

(b)  $\sim{u} = 3\tilde{i} + 2\tilde{k}$  and  $\sim{v} = 2\tilde{j} - \tilde{k}$

Solution:

$$\sim{u} \times \sim{v} = -4\tilde{i} + 3\tilde{j} + 7\tilde{k}$$

46. (9.4) Compute the following

- (a)  $\tilde{i} \times \tilde{j}$
- (b)  $\tilde{i} \times \tilde{i}$
- (c)  $\tilde{j} \times \tilde{k}$
- (d)  $\tilde{i} \times \tilde{k}$
- (e)  $\tilde{i} \times (\tilde{j} \times \tilde{i})$
- (f)  $\tilde{k} \times (\tilde{i} \times \tilde{j})$

Solution:

- (a)  $\tilde{k}$
- (b)  $\tilde{0}$
- (c)  $\tilde{i}$

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- (d)  $-\tilde{j}$
- (e)  $\tilde{j}$
- (f)  $\tilde{0}$

47. (9.4) Consider the triangle formed by the points  $P(1, 0, -3)$ ,  $Q(2, -1, 5)$  and  $R(0, 3, 1)$ .

(a) Find the length of the sides of the triangle.

Solution:

$$|PQ| = \sqrt{1 + 1 + 64} = \sqrt{66} \quad |QR| = \sqrt{4 + 16 + 16} = \sqrt{36}$$

$$|PR| = \sqrt{1 + 9 + 16} = \sqrt{26}$$

(b) Find the angle at Q.

Solution: If  $\theta$  is the angle at Q, then

$$\cos(\theta) = \frac{QP \cdot QR}{|QP| |QR|} = \frac{\langle -1, 1, -8 \rangle \cdot \langle -2, 4, -4 \rangle}{\sqrt{66} \sqrt{36}} = \frac{20}{6 \sqrt{66}} = \frac{10}{3 \sqrt{66}}$$

$$\theta = \arccos \frac{20}{3\sqrt{66}} \approx 0.608 \approx 34.85^\circ$$

(c) Find the area of the triangle.

Solution: The area of the triangle is half of the area of the parallelogram spanned by  $\vec{QP}$  and  $\vec{QR}$ . So

$$A = \frac{1}{2} |\vec{QP} \times \vec{QR}| = \frac{1}{2} | \langle 1 \cdot 4 - (-8)(-4), (-8)(-2) - (-4)(-1), (-1)4 - 1(-2) \rangle |$$

$$= \frac{1}{2} | \langle -28, 12, 2 \rangle | = \frac{1}{2} \sqrt{196 + 36 + 4} = \sqrt{233}$$

48. (9.4) To tighten a bolt a force of 12 N is applied to a wrench at a distance of 15 cm. The angle between the wrench and the applied force is  $30^\circ$ . What is the magnitude of the torque?

Solution:

$$|\vec{\tau}| = 15 \text{ cm} \cdot 12 \text{ N} \cdot \sin(30^\circ) = 90 \text{ cN m} = 0.9 \text{ N m}$$

49. (9.4) Find the volume of the parallelepiped spanned by  $\vec{a} = \langle 3, -1, 3 \rangle$ ,  $\vec{b} = \langle 1, 0, 4 \rangle$  and  $\vec{c} = \langle 0, -3, 1 \rangle$ .

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Solution:

$$V = |\vec{a} \cdot (\vec{b} \times \vec{c})| = | \langle 3, -1, 3 \rangle \cdot \langle 12, -1, -3 \rangle | = |36 + 1 - 9| = 28$$

The volume of the parallelepiped is 28.

50. (9.5) Find the vector equation and the parametric equations for the given lines.

(a) The line through  $P(3, -1, 12)$  parallel to the vector  $\langle 1, 0, 3 \rangle$ .

Solution: The vector equation is

$$\vec{r} = \langle 3, -1, 12 \rangle + t \langle 1, 0, 3 \rangle$$

and the parametric equations are

$$x = 3 + t \quad y = -1 \quad z = 12 + 3t$$

(b) The line through  $P(4, -1, 0)$  and  $Q(3, 2, 1)$ .

Solution: A possible direction vector is  $\langle 3 - 4, 2 - (-1), 1 - 0 \rangle = \langle -1, 3, 1 \rangle$ , so the vector equation is

$$\vec{r} = \langle 4, -1, 0 \rangle + t \langle -1, 3, 1 \rangle \quad \text{alternatively: } \vec{r} = \langle 3, 2, 1 \rangle + t \langle -1, 3, 1 \rangle$$

and the parametric equations are

$$x = 4 - t \quad y = -1 + 3t \quad z = t$$

51. (9.5) Find the vector equation and the scalar equation for the given planes.

(a) The plane containing  $P(3, 1, 6)$ ,  $Q(-2, 1, 4)$  and  $R(3, 1, -1)$ .

Solution: The vectors  $\vec{PQ}$  and  $\vec{PR}$  lie on the plane, so the normal vector is  $\vec{n} =$

$$\vec{PQ} \times \vec{PR} = \langle -5, 0, -2 \rangle \times \langle 0, 0, -7 \rangle = \langle 0, -35, 0 \rangle = (-35) \langle 0, 1, 0 \rangle$$

So the vector equation is

$$\langle 0, 1, 0 \rangle \cdot (\vec{r} - \langle 3, 1, 6 \rangle) = 0$$

and the scalar equation is

$$y - 1 = 0$$

(b) The plane containing  $P(6, 2, -2)$  and  $Q(1, -2, 0)$ , that is parallel to the vector  $\langle -1, 4, 2 \rangle$ .

Solution: The vector  $\vec{PQ}$  lies in the plane, so the normal vector is perpendicular to it. The normal vector also has to be perpendicular to  $\langle -1, 4, 2 \rangle$ , so

$$\begin{aligned} \vec{n} = \vec{PQ} \times \langle -1, 4, 2 \rangle &= \langle -5, -4, 2 \rangle \times \langle -1, 4, 2 \rangle = \langle -8 - 8, -2 + 10, -20 - 4 \rangle \\ &= \langle 0, 8, -24 \rangle = 8 \langle 0, 1, -3 \rangle \end{aligned}$$

So the vector equation is

$$\langle 0, 1, -3 \rangle \cdot (\vec{r} - \langle 6, 2, -2 \rangle) = 0$$

and the scalar equation is

$$y - 2 + (-3)(z + 2) = 0 \quad y - 3z = 8$$

(c) The plane perpendicular to  $\langle 2, -2, 5 \rangle$  containing  $P(0, 3, -2)$ .

Solution: The vector equation is

$$\langle 2, -2, 5 \rangle \cdot (\vec{r} - \langle 0, 3, -2 \rangle) = 0$$

and the scalar equation is

$$2x + (-2)(y - 3) + 5(z + 2) = 0 \quad 2x - 2y + 5z = -4$$

52. (9.5) Find the intersection of the given objects.

(a) The line  $\vec{r} = h3, 5, -4i + th2, -1, 0i$  and the line  $x = 3 + s, y = -2 - 4s, z = 2 + 3s$ .

Solution: To find the intersection, set the lines equal, that gives three

$$\text{equations } 3 + 2t = 3 + s \quad 5 - t = -2 - 4s \quad -4 = 2 + 3s$$

The third equation gives  $s = -2$ , plugged into the first two

$$3 + 2t = 3 - 2 \quad 5 - t = -2 + 8 \quad 2t = -2 - t = 1$$

Both are solved by  $t = -1$ . So the intersection point is  $(1, 6, -4)$ .

(b) The line  $\vec{r} = h0, 4, -1i + th3, -7, 7i$  and the plane  $3x - 5y + 2z = 10$

Solution: To parametric equations of the line are

$$x = 3t \quad y = 4 - 7t \quad z = -1 + 7t$$

Plugging these into the scalar equation of the plane gives

$$3(3t) - 5(4 - 7t) + 2(-1 + 7t) = 10 \quad 9t - 20 + 25t - 2 + 14t = 10$$

$$48t = 32 \quad t = \frac{32}{48} = \frac{2}{3}$$

So the point of intersection is  $(2, 4 - \frac{14}{3}, -1 + \frac{14}{3}) = (2, \frac{2}{3}, \frac{11}{3})$ .

(c) the plane  $\langle 4, 0, -2 \rangle \cdot (\vec{r} - \langle 1, 0, 5 \rangle) = 0$  and the plane  $-2x + 6y - z = 9$



Solution: The planes are not parallel, since the normal vectors  $\langle 4, 0, -2 \rangle$  and  $\langle -2, 6, -1 \rangle$  are not parallel. So the intersection is a line. The direction of the line is orthogonal to both normal vectors, so

$$\langle 4, 0, -2 \rangle \times \langle -2, 6, -1 \rangle = \langle 12, 8, 2 \rangle = 2 \langle 6, 4, 1 \rangle$$

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is the direction vector of the line of intersection. It remains to find one point, that lies on both planes. Consider the scalar equation of each plane

$$4x - 2z = -6 - 2x + 6y - z = 9$$

Setting  $x = 0$ , gives  $z = 3$  and  $y = 2$ . So  $(0, 2, 3)$  lies on both planes and the line of intersection is

$$\vec{r} = \langle 0, 2, 3 \rangle + t \langle 6, 4, 1 \rangle$$

53. (9.5) Find the angle in which the given planes intersect.

(a)  $\langle 1, -3, 0 \rangle \cdot (\vec{r} - \langle 1, 0, 4 \rangle) = 0$  and  $\langle 0, -2, 4 \rangle \cdot (\vec{r} - \langle 5, -2, 0 \rangle) = 0$

Solution: The planes are clearly not parallel, so they intersect in a line. The angle in which they intersect is the same as the angle between the normal vectors. So

$$\cos(\theta) = \frac{\langle 1, -3, 0 \rangle \cdot \langle 0, -2, 4 \rangle}{\|\langle 1, -3, 0 \rangle\| \|\langle 0, -2, 4 \rangle\|}$$

$$= \frac{6}{\sqrt{10} \sqrt{20}}$$

$$= \frac{6}{\sqrt{10} \cdot \sqrt{4 \cdot 5}} = \frac{6}{\sqrt{10} \cdot 2\sqrt{5}} = \frac{3}{\sqrt{50}} = \frac{3}{5\sqrt{2}}$$

$$= \frac{3\sqrt{2}}{5\sqrt{2} \cdot \sqrt{2}} = \frac{3\sqrt{2}}{10}$$

$$\theta = \arccos\left(\frac{3\sqrt{2}}{10}\right) \approx 1.13 = 64.9^\circ$$

$$\theta = \arccos\left(\frac{3\sqrt{2}}{10}\right)$$

So the planes intersect at an angle of  $64.9^\circ$ .

(b)  $2x - 7y + 3z = 12$  and  $\langle 1, 5, 2 \rangle \cdot (\vec{r} - \langle 5, -6, 2 \rangle) = 0$

Solution: The planes are clearly not parallel, so they intersect in a line. The angle in which they intersect is the same as the angle between the normal vectors. So

$$\begin{aligned}
 & | \langle 2, -7, 3 \rangle \cdot \langle 1, 5, 2 \rangle | = 2 - 35 + 6 \\
 \cos(\theta) &= \frac{\langle 2, -7, 3 \rangle \cdot \langle 1, 5, 2 \rangle}{\sqrt{4 + 49 + 9} \sqrt{1 + 25 + 4}} = \frac{-27}{\sqrt{62} \sqrt{30}} \\
 & \approx 2.25 = 128.8^\circ \\
 \theta &= \arccos \left( \frac{-27}{\sqrt{62} \sqrt{30}} \right)
 \end{aligned}$$

So the planes intersect at an angle of  $51.2^\circ$  (the angle of intersection between two planes can always be chosen  $\leq 90^\circ$ ).

54. (9.5) Derive the equation to find the point  $\tilde{x}$  in a plane  $\tilde{n} \cdot (\tilde{x} - \tilde{p}) = 0$  that is closest to a given point in space  $\tilde{y}$  is given by the formula in terms of the plane normal vector  $\tilde{n}$  and plane reference point  $\tilde{p}$ . Also find the distance formula between the points.

Solution: The minimum-distance point  $\tilde{x}$  is found when  $\tilde{x} - \tilde{y} = A\tilde{n}$  for some  $A$  scalar. So, we need to find  $A$ . That is, the vectors are collinear, but may differ by a constant. Solving for  $\tilde{x}$ :

$$\tilde{x} = \tilde{y} + A\tilde{n}$$

$\tilde{x}$  needs to also be in the plane, so

$$0 = \tilde{n} \cdot (\tilde{x} - \tilde{p}) = \tilde{n} \cdot (\tilde{y} + A\tilde{n} - \tilde{p}),$$

so

$$\tilde{n} \cdot \tilde{y} + A\|\tilde{n}\|^2 - \tilde{n} \cdot \tilde{p} = 0$$

Solving for  $A$ :

$$A = \frac{\tilde{n} \cdot \tilde{p} - \tilde{n} \cdot \tilde{y}}{\|\tilde{n}\|^2},$$

which yields the formula

$$(\tilde{y} - \tilde{p}) \cdot \tilde{n}$$

$$\tilde{x} = \tilde{y} - \frac{\tilde{n} \cdot (\tilde{y} - \tilde{p})}{\|\tilde{n}\|^2} \tilde{n}$$

and the distance equation is

$$\|\tilde{x} - \tilde{y}\| = \frac{|(\tilde{y} - \tilde{p}) \cdot \tilde{n}|}{\|\tilde{n}\|}$$

55. (9.5) Find the distance of the given objects.

(a) The plane  $3x - 4z = 25$  and the point  $(2, 6, 3)$

Solution: A point  $\tilde{x}$  in the plane that is closest to a given point in space  $\tilde{y}$  is given by the formula in terms of the plane normal vector  $\tilde{n}$  and plane reference point  $\tilde{p}$ :

$$\tilde{x} = \tilde{y} - \frac{(\tilde{y} - \tilde{p}) \cdot \tilde{n}}{\|\tilde{n}\|^2} \tilde{n}$$

and the distance equation is

$$\|\tilde{x} - \tilde{y}\| = \frac{|(\tilde{y} - \tilde{p}) \cdot \tilde{n}|}{\|\tilde{n}\|}$$

To use the formula for the distance between a point and a plane, one needs the scalar equation of the plane. For this plane the scalar equation is

$$3x - 4z + 25 = 0$$

So the distance is

$$d = \frac{|3 \cdot 2 - 4 \cdot 3 + 25|}{\sqrt{3^2 + 4^2}} = \frac{19}{5}$$

(b) The plane  $5x - 8y + 2z = 2$  and the line  $\tilde{r} = 2\mathbf{i} - 4\mathbf{j} + 3t\mathbf{i} + 4t\mathbf{j} - 2t\mathbf{k}$

Solution: To see plane and the line are parallel, check if the normal vector of the plane and the direction vector of the line are orthogonal

$$\langle 5, -8, 2 \rangle \cdot \langle 4, 2, -2 \rangle = 20 - 16 - 4 = 0$$

So they are orthogonal. Now the distance of the line and the plane is the same as the distance from any point on the line and the plane. A point on the line is  $(2, -4, 3)$ . The distance of the plane to this point is

$$d = \frac{|5 \cdot 2 - 8 \cdot (-4) + 2 \cdot 3 - 2|}{\sqrt{25 + 64 + 4}} = \frac{46}{9}$$

This is also the distance of the line and the plane.

(c) the plane  $2x - 6y + 2z = -11$  and the plane  $\langle -1, 3, -1 \rangle \cdot (\sim r - \langle -2, 3, 5 \rangle) = 0$  Page 44

Solution: First check, if the planes are parallel. The normal vectors  $\langle 2, -6, 2 \rangle$  and  $\langle -1, 3, -1 \rangle$  are clearly multiples of each other. So the planes are parallel. Now the distance between the two planes is the same as the distance of any point on one plane to the other plane. So it is enough to find the distance of  $(-2, 3, 5)$ , a point on the second plane, to the first plane. The distance between these is

$$d = |2(-2) - 6 \cdot 3 + 2 \cdot 5 + 11|$$

$$\sqrt{4 + 36 + 4} = \sqrt{44} = 2\sqrt{11}$$

This is the distance between the two planes.

(d) the line  $\sim r = \langle -3, 0, 7 \rangle + t \langle 3, -2, 1 \rangle$  and the line  $x = 3t - 2, y = -t + 2, z = 11$

Solution: First construct a plane, that contains the first line and is parallel to the second line. The normal vector  $\sim n$  of this plane has to be orthogonal to the direction vector of both planes, which are  $\langle 3, -2, 1 \rangle$  and  $\langle 3, -1, 0 \rangle$ , so take

$$\sim n = \langle 3, -2, 1 \rangle \times \langle 3, -1, 0 \rangle = \langle 1, 3, 3 \rangle$$

Then the plane is

$$\langle 1, 3, 3 \rangle \cdot (\sim r - \langle -3, 0, 7 \rangle) = 0 \quad x + 3y + 3z + 18 = 0$$

Now the distance between the two lines is the distance of this plane to a point on the second line. Take  $(-2, 2, 11)$  as a point on the second line, then the distance is

$$d = |1(-2) + 3(2) + 3(11) + 18|$$

$$\sqrt{1 + 9 + 9} = \sqrt{19}$$

This is the distance between the two lines.

56. (9.5) Decide whether the given lines are the same, parallel, intersect or neither. (a)  $L_1: \sim r = \langle 2, 4, -1 \rangle + t \langle 3, 4, 1 \rangle$  and  $L_2: y = 2, \frac{x-2}{3} = \frac{z-1}{-1}$

Solution: The direction vector of  $L_2$ , which is given by symmetric equations, is  $\langle 3, 0, -1 \rangle$ . So the lines are neither the same nor parallel. To check, if they intersect, look at the parametric equations of  $L_1$

$$x = 2 + 3t \quad y = 4 + 4t \quad z = -1 + t$$

Now for  $L_2$ , it is  $y = 2$ , so  $t = \frac{1}{2}$ . This gives the point  $(\frac{7}{2}, 2, -\frac{1}{2})$ . Now check, if this point is on  $L_2$ :

$$\frac{3}{2} = \frac{3}{2} = \frac{-1}{2} = -\frac{1}{2}$$

So the lines intersect.

(b)  $L_1: \vec{r} = \langle 3, -1, 2 \rangle + t \langle 3, -6, 9 \rangle$  and  $L_2: \vec{r} = \langle 0, 5, -7 \rangle + t \langle -1, 2, -3 \rangle$

Solution: The lines are the same or parallel, since the direction vectors are multiples of each other  $\langle 3, -6, 9 \rangle = (-3) \langle -1, 2, -3 \rangle$ . To check, if they are equal, just check, if a point, that lies on  $L_1$ , also lies in  $L_2$ . Take  $P(3, -1, 2)$ . Now solve the equations

$$3 = -t - 1 = 5 + 2t \quad 2 = -7 - 3t$$

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$t = -3$  solves all these equations simultaneously, so the lines are the same.

(c)  $L_1: x = 2 + t, y = -3 + 2t, z = -5t$  and  $L_2: \vec{r} = \langle 1, 0, 2 \rangle + t \langle -2, -4, 10 \rangle$

Solution: A direction vector of the first line is  $\langle 1, 2, -5 \rangle$ . Since  $\langle -2, -4, 10 \rangle = (-2) \langle 1, 2, -5 \rangle$ , the lines are parallel or the same. Now check, if the point  $P(2, -3, 0)$ , which lies on  $L_1$ , also lies on  $L_2$ . That is solve the equations

$$2 = 1 - 2t \quad -3 = -4t \quad 0 = 2 + 10t$$

The point does not lie on  $L_2$ , since the first equation is solved by  $t = \frac{1}{2}$  and the second by  $t = \frac{3}{4}$ . So the lines are parallel.

57. (9.5) Decide whether the given planes are the same, parallel or intersect.

(a) Plane 1:  $\langle 1, 0, -3 \rangle \cdot (\vec{r} - \langle 2, 0, 3 \rangle) = 0$  and plane 2:  $2x - 3y + 2z + 4 = 0$

Solution: The normal vector of the first plane is  $\langle 1, 0, -3 \rangle$  and the normal vector of the second plane is  $\langle 2, -3, 2 \rangle$ . These are not parallel, so the planes are neither parallel nor the same. So they intersect.

(b) Plane 1:  $-4x + 3y - z = 12$  and plane 2:  $\langle 12, -9, 3 \rangle \cdot (\sim r - \langle -2, 1, -1 \rangle) = 0$

Solution: The normal vector of the first plane is  $\langle -4, 3, -1 \rangle$  and the normal vector of the second plane is  $\langle 12, -9, 3 \rangle$ . These vectors are multiples of each other:  $\langle 12, -9, 3 \rangle = (-3) \langle -4, 3, -1 \rangle$ . So the planes are parallel or the same.  $\langle -2, 1, -1 \rangle$  is a point in the second plane, plugging it into the equation of the first plane

$$-4(-2) + 3 \cdot 1 - (-1) = 8 + 3 + 1 = 12$$

So the point also lies in the first plane. So the planes are the same.

(c) Plane 1:  $8x - 4y = 2z - 2$  and plane 2:  $-4x + 2y + z + 14 = 0$

Solution: The normal vector of the first plane is  $\langle 8, -4, -2 \rangle$  and the normal vector of the second plane is  $\langle -4, 2, 1 \rangle$ . These are multiples of each other:  $\langle 8, -4, -2 \rangle = (-2) \langle -4, 2, 1 \rangle$ . So the planes are parallel or the same. A point on the first plane is  $(0, 0, 1)$ . Plugging this into the second equation

$$-4 \cdot 0 + 2 \cdot 0 + 1 + 14 = 15 \neq 0$$

so the point does not lie in the plane. So the planes are parallel.

58. (9.6) Consider the surface  $4x^2 + y^2 + 4z^2 = 16$ .

(a) Draw the traces in  $x = k$  for three different values for  $k$ .

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z

2

y

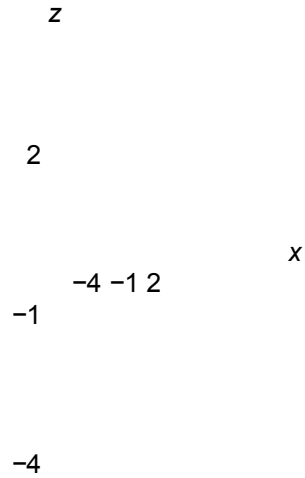
-4 -1 2

-1

-4

Solution: The outer trace is for  $k = 0$ , the inner for  $k = \pm 1$ . If  $-2 > k$  or  $k > 2$ , the equation has not solution.

(b) Draw the traces in  $y = k$  for three different values for  $k$ .



Solution: The outer trace is for  $k = 0$ , the inner for  $k = \pm 1$ . If  $-4 > k$  or  $k > 4$ , the equation has not solution.

(c) Draw the traces in  $z = k$  for three different values for  $k$ .

$$-1 \quad -4 \quad -1 \quad 2 \quad x$$

$$-4$$

Solution: If  $-2 > k$  or  $k > 2$ , the equation has not solution.

(d) Describe and or sketch the shape of the surface in three dimensions.

Solution: The shape is an ellipsoid (an egg), with the long axis on the  $y$ -axis.

59. (9.7) Each point is given in either rectangular, cylindrical or spherical coordinate. Fill in the missing coordinates.

Rectangular  $(x, y, z)$    Cylindrical  $(r, \theta, z)$    Spherical  $(\rho, \theta, \varphi)$

$(2, 2, 0)$     $(2, \sqrt{2}, \frac{\pi}{4}, \frac{\pi}{2})$

$(2\sqrt{2}, \frac{\pi}{4}, 0)$
$(0, 0, 3)$ $\sqrt{\quad}$

$(0, 0, 3)$     $(3, 0, 0)$

$(1, -1, 1)$     $(2, \frac{7\pi}{4}, 1)$     $(\sqrt{3}, \frac{7\pi}{4}, \arccos \frac{1}{\sqrt{3}} \approx 0.96 \approx 54.7^\circ)$

60. (9.7) Consider surface the cone with the tip at the origin and who's axis goes along the negative  $z$ -axis down to a base with radius radius 1 and height 2 from the tip.

(a) Find the equation describing the cone in rectangular coordinates.

Solution:

$$x^2 + y^2 = \left(\frac{z}{2}\right)^2, 0 \leq z \leq 2$$

(b) Find the equation describing the cone in cylindrical coordinates.



Solution:

$$r = -z/2$$

(c) Find the equation describing the cone in spherical coordinates.

Solution:

$$0 \leq \rho \leq \sqrt{1 + 4\phi} = \frac{5\pi}{6}$$

$\theta$  is free to vary.

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61. (10.1) Sketch the curve of the given vector equation.

(a)  $\vec{r}(t) = h \sin(t), 3 \cos(t), 2t$ .

(b)  $\vec{r}(t) = ht, -1, t^2i$ .

(c)  $\vec{r}(t) = h \cos(t), \sin(t), \cos(t)i$ .

Solution:

(a) An ellipse in the plane  $z = 2$ , with  $y$ -diameter 3 and  $x$ -diameter 1.

(b) An upward u-shaped parabola lying in the  $y = -1$  plane where  $z = x^2$ .

(c) An ellipse lying on a slanted plane with  $y$ -diameter 1 and  $x$ -diameter 1. The normal vector of the plane can be computed as

$$\begin{aligned} \vec{n} &= \vec{r}' \times \vec{r}'' \\ \det \begin{vmatrix} i & j & k \\ h \cos(t) & \sin(t) & -\sin(t) \\ -h \sin(t) & \cos(t) & -\cos(t) \end{vmatrix} &= h-1, 0, 1i. \end{aligned}$$

62. (10.1) Specify a vector function that draws the following described paths

(a) A circle in the  $x$ - $y$  plane ( $z = 0$ ).

(b) Any spiral emanating from the origin and expanding outward that lies in the  $y$ - $z$  plane (c) A helix that begins in the  $x$ - $y$  plane at point  $(0, -1, 0)$  at  $t = 0$  and spirals upward, drawing a circle with radius 1 when observed looking down along the  $z$ -axis onto the  $x$ - $y$  plane.

Solution: These are examples. Other formulas could work as well.

(a)  $\vec{r}(t) = h\cos(t), \sin(t), 0i$

(b)  $\vec{r}(t) = h0, t \cos(t), t\sin(t)i$

(c)  $\vec{r}(t) = h\sin(t), -\cos(t), ti$

63. (10.1) Harder: Find the vector function, that represents the curve of intersection of the given surfaces.

(a)  $x^2 + z^2 = y^2$  and  $x + 2y = 1$

Solution: Because of the first equation, set

$$x = y \cos(t) \quad z = y \sin(t)$$

Plugging this into the second one gives

$$y \cos(t) + 2y = 1 \quad y = \frac{1}{2 + \cos(t)}$$

So the vector function for the intersection is

$$\vec{r}(t) = \left\langle \frac{\cos(t)}{2 + \cos(t)}, \frac{1}{2 + \cos(t)}, \frac{\sin(t)}{2 + \cos(t)} \right\rangle \quad 0 \leq t \leq 2\pi$$

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(b)  $x^2 - y^2 + z = 1$  with  $z \geq 0$  and  $2y - z = 1$

Solution: Solving the second for  $z$  gives  $z = 2y - 1$ , plugging this into the first equation

$$x^2 - y^2 + 2y = 1 \quad x^2 - (y - 1)^2 = 0 \quad y \geq \frac{1}{2}$$

So it is  $y = \sqrt{x^2 - 1} + 1$ . Set  $t = x$ , so the vector function of the intersection is

$$\vec{r}(t) = \left\langle t, \sqrt{t^2 - 1} + 1, 2\sqrt{t^2 - 1} + 1 \right\rangle$$

64. (10.2) Find the tangent line for the given vector functions at the given points.

(a)  $\vec{r}(t) = \langle t^2, 2t, t^3 \rangle$  at  $(1, 2, 1)$

Solution: The tangent vector is

$$\sim r'(t) = \langle 2t, 2, 3t^2 \rangle \sim r'(1) = \langle 2, 2, 3 \rangle$$

Since the  $t$ -value at  $(1, 2, 1)$  is  $t = 1$ . So the tangent line is

$$x = 1 + 2t \quad y = 2 + 2t \quad z = 1 + 3t$$

(b)  $\sim r(t) = \frac{1}{t} \sim i - \ln(t) \sim j + \sqrt{t} \sim k$  at  $(1, 0, 1)$

Solution: The tangent vector is

$$\sim r'(t) = -\frac{1}{t^2} \sim i - \frac{1}{t} \sim j + \frac{1}{2\sqrt{t}} \sim k \quad \sim r'(1) = -\sim i - \sim j + \frac{1}{2} \sim k$$

Since the  $t$ -value at  $(1, 0, 1)$  is  $t = 1$ . So the tangent line is

$$x = 1 - t \quad y = -t \quad z = 1 + \frac{1}{2}t$$

65. (10.2) Given two vector function  $\sim u(t)$ ,  $\sim v(t)$  and  $\sim w(t)$  with

$$\begin{aligned} \sim u(1) &= \langle 3, -1, 3 \rangle \quad \sim u'(1) = \langle -2, 0, 6 \rangle \quad \sim v(1) = \langle 0, 2, 1 \rangle \quad \sim v'(1) = \langle 9, 1, 3 \rangle \quad \sim w(1) = \langle \\ &1, 0, 3 \rangle \quad \sim w'(1) = \langle 2, 4, 1 \rangle \end{aligned}$$

Evaluate the following expressions.

(a)  $\frac{d}{dt}(\sim u(t) \cdot \sim v(t))$  at  $t = 1$

Solution:

$$_{t=1} = (\sim u'(t) \cdot \sim v(t) + \sim u(t) \cdot \sim v'(t))_{t=1} = 6 + (18 - 1 + 9) = 32$$

$\frac{d}{dt}$

$$\frac{d}{dt}(\sim u(t) \cdot \sim v(t))$$

(b)  $\frac{d}{dt}(\sim v(t) \times \sim u(t))$  at  $t = 1$

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Solution:

$$_{t=1} = \sim v'(1) \times \sim u(1) + \sim v(1) \times \sim u'(1) = \langle 18, -10, -8 \rangle$$

$\frac{d}{dt}$

$$\frac{d}{dt}(\sim v(t) \times \sim u(t))$$

(c)  $\frac{d}{dt}(\tilde{w}(t) \cdot (\tilde{v}(t) \times \tilde{u}(t)))$  at  $t = 1$

Solution:

$$\frac{d}{dt}(\tilde{w}(t) \cdot (\tilde{v}(t) \times \tilde{u}(t))) = \tilde{w}'(1) \cdot (\tilde{v}(1) \times \tilde{u}(1)) + \tilde{w}(1) \times \frac{d}{dt}(\tilde{v}(t) \times \tilde{u}(t)) \quad t=1$$

$$= \langle 2, 4, 1 \rangle \cdot \langle 7, 3, -6 \rangle + \langle 1, 0, 3 \rangle \cdot \langle 18, -10, -8 \rangle$$

$$= 14 + 12 - 6 + 18 - 24 = 14$$

66. (10.3) Find the arc length of the given curves.

(a)  $\tilde{r}(t) = \langle 1 - 2t^2, t^3, -t^3 \rangle$  for  $0 \leq t \leq \sqrt{2}$

Solution:

$$|\tilde{r}'(t)| = \sqrt{(-4t)^2 + (3t^2)^2 + (-3t^2)^2} = \sqrt{16t^2 + 9t^4 + 9t^4} = \sqrt{16t^2 + 18t^4}$$

$$L = \int_0^{\sqrt{2}} \sqrt{16t^2 + 18t^4} dt$$

with substitution  $u = 16 + 18t^2, du = 36tdt$ .

(b)  $\tilde{r}(t) = e^t \mathbf{i} - 2t \mathbf{j} + e^{-t} \mathbf{k}$  for  $-1 \leq t \leq 1$

Solution:

$$|\tilde{r}'(t)| = \sqrt{(e^t)^2 + (-2)^2 + (-e^{-t})^2} = \sqrt{e^{2t} - 2 + e^{-2t}} = \sqrt{(e^t - e^{-t})^2} = e^t - e^{-t}$$

$$L = \int_{-1}^1 (e^t - e^{-t}) dt = [e^t + e^{-t}]_{-1}^1 = (e + e^{-1}) - (e^{-1} + e) = 0$$

67. (10.3) Find the parametrization with respect to arc length measured from the point where  $t = 0$ .

(a)  $\tilde{r}(t) = (1 + 3t)\tilde{i} - t\tilde{j} + (2 - t)\tilde{k}$

Solution: First find the arc length function

$$\tilde{r}'(t) = 3\tilde{i} - \tilde{j} - \tilde{k} \quad |\tilde{r}'(t)| = \sqrt{11}$$

$$s(t) = \int_0^t |\tilde{r}'(u)| du = \int_0^t \sqrt{11} du = \sqrt{11}t$$

So the reparametrization is

$$\tilde{r}(t(s)) = (1 + 3s)\sqrt{11}\tilde{i} - s\sqrt{11}\tilde{j} + (2 - s)\sqrt{11}\tilde{k}$$

(b)  $\tilde{r}(t) = \langle 1, e^t \sin(t), -e^t \cos(t) \rangle$

Solution: First find the arc length function

$$\tilde{r}'(t) = \langle 0, e^t \sin(t) + e^t \cos(t), -e^t \cos(t) + e^t \sin(t) \rangle$$

$$|\tilde{r}'(t)| = \sqrt{e^{2t}(\sin(t) + \cos(t))^2 + e^{2t}(-\cos(t) + \sin(t))^2} = e^t \sqrt{2}$$

$$s(t) = \int_0^t e^u \sqrt{2} du = \sqrt{2}e^t - 1$$

Solving this for  $t$  gives

$$\frac{s}{\sqrt{2}} + 1 = e^t \Rightarrow t = \ln \left( \frac{s}{\sqrt{2}} + 1 \right)$$

So the reparametrization is

$$\tilde{r}(t(s)) = \langle 1, \left( \frac{s}{\sqrt{2}} + 1 \right) \sin \ln \left( \frac{s}{\sqrt{2}} + 1 \right), - \left( \frac{s}{\sqrt{2}} + 1 \right) \cos \ln \left( \frac{s}{\sqrt{2}} + 1 \right) \rangle$$

68. (10.3) Find the curvature of the given curves and the specified points.

(a)  $\tilde{r}(t) = \langle e^{2t}, 3 - t^2, t \rangle$  at  $(1, 3, 0)$

Solution:

$$\tilde{r}^0(t) = \langle 2e^{2t}, -2t, 1 \rangle \quad \tilde{r}^{00}(t) = \langle 4e^{2t}, -2, 0 \rangle$$

$$\tilde{r}^0(t) \times \tilde{r}^{00}(t) = \langle 2, 4e^{2t}, -4e^{2t} + 8te^{2t} \rangle$$

$$|\tilde{r}^0(t)| = \sqrt{4e^{4t} + 4t^2 + 1} \quad |\tilde{r}^0(t) \times \tilde{r}^{00}(t)| = \sqrt{4 + 16e^{4t} + 16e^{4t}(1 - 4t + 4t^2) + 16e^{4t} + 16e^{4t}(1 - 4t + 4t^2)}$$

$$\kappa(t) = \frac{|\tilde{r}^0(t) \times \tilde{r}^{00}(t)|}{|\tilde{r}^0(t)|^3} = \frac{\sqrt{4e^{4t} + 4t^2 + 1}}{(4e^{4t} + 4t^2 + 1)^{3/2}}$$

$$|\tilde{r}^0(t)|^3 =$$

The  $t$ -value at  $(1, 3, 0)$  is  $t = 0$ . So the curvature is

$$\frac{\sqrt{4 + 16 + 16}}{(4 + 1)^{3/2}} = \frac{5}{5} = 1$$

$$\kappa(0) = 1$$

(b)  $\tilde{r} = t \sin(t) \tilde{i} + 4t \tilde{j} + t \cos(t) \tilde{k}$  at  $\frac{\pi}{2}, 2\pi, 0$

Solution:

$$\tilde{r}^0(t) = (t \cos(t) + \sin(t)) \tilde{i} + 4 \tilde{j} + (-t \sin(t) + \cos(t)) \tilde{k}$$

$$\tilde{r}^{00}(t) = (-t \sin(t) + 2 \cos(t)) \tilde{i} + (-t \cos(t) - 2 \sin(t)) \tilde{k}$$

$$\tilde{r}^0(t) \times \tilde{r}^{00}(t) = 4(-t \cos(t) - 2 \sin(t)) \tilde{i} + (t^2 + 2) \tilde{j} + (-4)(-t \sin(t) + \cos(t)) \tilde{k}$$

$$|\tilde{r}^0(t)| = \sqrt{t^2 + 1} \quad |\tilde{r}^0(t) \times \tilde{r}^{00}(t)| = \sqrt{16t^2 + 64 + t^4 + 2t^2 + 4}$$

$$\kappa(t) = \frac{\sqrt{t^4 + 17t^2 + 68}}{(t^2 + 1)^{3/2}}$$

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The  $t$ -value at  $\frac{\pi}{2}, 2\pi, 0$  is  $t = \frac{\pi}{2}$ . So the curvature is

$$\kappa = \frac{\sqrt{4 + 17 \left(\frac{\pi}{2}\right)^2 + 68}}{\left(\left(\frac{\pi}{2}\right)^2 + 1\right)^{3/2}}$$

$$\frac{2}{2} \sqrt{\frac{\pi}{2}}$$

69. (10.3) Find the unit tangent vector, the unit normal vector and the binormal vector of the given curves.

(a)  $\tilde{r}(t) = t \tilde{i} - t^2 \tilde{k}$

Solution:

$$\tilde{r}'(t) = \tilde{i} - 2t \tilde{k} \quad |\tilde{r}'(t)| = \sqrt{1 + 4t^2}$$

$$\tilde{T}(t) = \frac{1}{\sqrt{1 + 4t^2}} (\tilde{i} - 2t \tilde{k})$$

$$\tilde{T}'(t) = \frac{-8t}{(1 + 4t^2)^{3/2}} \tilde{i} + \frac{2}{(1 + 4t^2)^{3/2}} \tilde{k}$$

$$|\tilde{T}'(t)| = \frac{2\sqrt{1 + 4t^2}}{(1 + 4t^2)^{3/2}} = \frac{2}{1 + 4t^2}$$

$$\tilde{N}(t) = \frac{1}{2} (\tilde{i} - 2t \tilde{k})$$

$$\tilde{B}(t) = \tilde{j}$$

(b)  $\tilde{r}(t) = \langle \cos(3t), 4t, \sin(3t) \rangle$

Solution:

$$\tilde{r}'(t) = \langle -3 \sin(3t), 4, 3 \cos(3t) \rangle \quad |\tilde{r}'(t)| = \sqrt{9 + 16} = 5$$

$$\tilde{T}(t) = \left\langle -\frac{3}{5} \sin(3t), \frac{4}{5}, \frac{3}{5} \cos(3t) \right\rangle$$

$$\tilde{T}'(t) = \langle -3 \cos(3t), 0, -5 \sin(3t) \rangle \quad |\tilde{T}'(t)| = 5$$

$$\tilde{N}(t) = \langle -\cos(3t), 0, -\sin(3t) \rangle$$

$$\tilde{B}(t) = \tilde{T}(t) \times \tilde{N}(t) = \langle -\frac{4}{5} \sin(3t), \frac{3}{5}, \frac{4}{5} \cos(3t) \rangle$$

70. (10.4) Consider the vector function  $\tilde{r}(t) = h \sin(t), \cos(t), ti$ . Find the decomposition of the acceleration into the osculating plane components:

$$\tilde{a} = a_T(t) \tilde{T}(t) + a_N(t) \tilde{N}(t)$$

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Solution:

$$\tilde{r}'(t) = h \cos(t), -\sin(t), 1i$$

$$\tilde{a}(t) = \tilde{r}''(t) = h - \sin(t), -\cos(t), 0i$$

$$\tilde{T}(t) = \frac{1}{\sqrt{h^2 + 1}} \langle h \cos(t), -\sin(t), 1 \rangle$$

$$a_T(t) = \tilde{T}(t) \cdot \tilde{a}(t) = 0.$$

$$\tilde{N}(t) = h - \sin(t), -\cos(t), 0i$$

$$a_N(t) = \tilde{N}(t) \cdot \tilde{a}(t) = -1$$

$$\tilde{a}(t) = a_N \tilde{N}(t).$$

71. (10.4) Consider  $\tilde{x} = h1, 2, 3i$  and an orthogonal coordinate system  $\tilde{u} = h1, 1, -1i$ ,  $\tilde{v} = h0, 1, 1i$ ,  $\tilde{w} = h2, -1, 1i$ . Find the triplet  $(a, b, c)$  such that

$$\tilde{x} = a\tilde{u} + b\tilde{v} + c\tilde{w}.$$



Solution:

$$\sim x \cdot \sim u = -1$$

$$\sim x \cdot \sim v = 5$$

$$\sim x \cdot \sim w = 3$$

$$|\sim u|^2 = 3$$

$$|\sim v|^2 = 2$$

$$|\sim w|^2 = 6$$

$$-1$$

$$5, 1$$

$$3, 2, 2 \cdot$$

$$(a, b, c) =$$

72. On an  $x$ - $y$  plane, draw precise level curves of the function  $f(x, y) = e^{-x^2-y^2} = z$  at the values  $f(x, y) = k$ , where  $k = e^{-1}, e^{-1/4}, e^{-1/16}$ . Describe the shape of the surface.

Solution:  $e^{-x^2-y^2} = e^{-1} \implies x^2 + y^2 = 1$ , which is a circle of radius 1 centered at the origin. The remaining two level curves are circles of radius 2 and 4. The surface looks like a mountain with summit located at the origin.

73. Draw precise traces and level curves of the function  $f(x, y) = x^2 - 4y^2 = z$  at the values near the origin
- (a)  $x = 0$ .
  - (b)  $y = 0$
  - (c)  $f(x, y) = 1$
  - (d)  $f(x, y) = -1$

and then describe the shape of the surface near the origin.

Solution: The surface is shown in the figure. The traces and level curves that are embedded in the surface are found as follows:

(a)  $z = -4y^2$  dark blue downward parabola.

(b)  $z = x^2$  dark red upward parabola.

(c)  $x = \pm \sqrt{1 + 4y^2}$  light blue hyperbola pairs.

(d)  $y = \pm \frac{1}{2} \sqrt{1 + x^2}$  magenta hyperbola pairs.

The surface resembles a mountain pass.

74. (11.1) Draw a contour map (i.e., level curves) for the function

$$f(x, y) = xy^2$$

4

y

-4 -2 2 4 x

-2

-4

75. Compute the two limit paths for the function  $|x|-|y|$

$$|x+|y| = z.$$

(a) The path  $y = 0$  and  $x \rightarrow 0^+$  to the origin.

(b) The path  $x = 0$  and  $y \rightarrow 0^+$  to the origin.

and determine if  $\lim_{(x,y) \rightarrow (0,0)} |x|-|y|$

$|x+|y|$  exists.

Solution:

(a)  $\lim_{y=0, x \rightarrow 0^+} |x|-|y|$

$$|x|+0 = 1.$$

(b)  $\lim_{x=0, y \rightarrow 0^+} |x|-|y|$

$$0+|y| = -1.$$

The limits on the two paths do not agree, so the limit as  $(x, y) \rightarrow (0, 0)$  cannot exist.

76. (11.2) Show that the following limits do not exist.

$$(a) \lim_{(x,y) \rightarrow (0,0)} \frac{\sin(x)^2 + y}{x^2 + y^2}$$

Solution: Set  $f(x, y) = \frac{\sin(x)^2 + y}{x^2 + y^2}$ . Along the x-axis it is

$$f(x, 0) = \frac{\sin(x)^2}{x^2} \xrightarrow{x \rightarrow 0} 1$$

Along the y-axis it is

$$f(0, y) = \frac{y}{y^2} = \frac{1}{y} \xrightarrow{y \rightarrow 0} \infty$$

which diverges. So the limit does not exist.

$$(b) \lim_{(x,y) \rightarrow (0,0)} \frac{x^2 + y^3}{xy}$$

Solution: Set  $f(x, y) = \frac{x^2 + y^3}{xy}$ . Along the x-axis it is

$$f(x, 0) = \frac{x^2}{0} \xrightarrow{x \rightarrow 0} \infty$$

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Along the diagonal  $x = y$  it is

$$f(x, x) = \frac{x^2 + x^3}{x^2} = 1 + x \xrightarrow{x \rightarrow 0} 1$$

Since these limits are different, the limit does not exist.

77. Determine if the following limits exist or not. If so, compute them; if not, explain why.

(a)  $\lim_{(x,y) \rightarrow (0,0)} \frac{\ln(x+y)}{2+x+y}$

(b)  $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2+2y+3}{2+x+y}$

(c)  $\lim_{(x,y) \rightarrow (1,3)} \frac{x+y}{6x-2y}$

(d)  $\lim_{(x,y) \rightarrow (1,3)} \frac{\sin(\pi y)x^2-y+2}{x^2-2y+3}$

Solution:

(a) DNE because the numerator is not defined at the origin.

(b) The function is rational with no singularity at the origin, so the function is continuous and the limit exists  $L = 3/2$ .

(c) DNE because the denominator is zero at the origin while the numerator is not zero.

(d) The function is continuous because it is comprised of continuous component functions. The limit is  $L = -1/(1 - 6 + 3) = 1/2$ .

78. Compute the partial derivatives

(a)  $\frac{\partial}{\partial x} [x \cos(xe^y)]$

(b)  $\frac{\partial}{\partial y} [x \cos(xe^y)]$

(c) (d)  $x^2 e^{xy}$   $xy e^{xy}$

Solution: Wolfram alpha should do it.

79. (11.3) Given the function

$$f(x, y) = xy^3 + \sin(xy) - e^{x^2}$$

Find the following partial derivatives.

(a)  $f_x(x, y)$

Solution:

$$f_x(x, y) = y^3 + y \cos(xy) - 2xe^{x^2}$$

(b)  $\frac{\partial f}{\partial y}$

$(x, y)$

Solution:  $\frac{\partial f}{\partial y}$

$$\frac{\partial f}{\partial y}(x, y) = 3xy^2 + x \cos(xy)$$

(c)  $f_{xy}(x, y)$

Solution:

$$f_{xy}(x, y) = 3y^2 + \cos(xy) - xy \sin(xy)$$

(d)  $\frac{\partial^2 f}{\partial x^2}$

Solution:  $\frac{\partial^2 f}{\partial x^2}$

$$\frac{\partial^2 f}{\partial x^2} = 6xy - x^2 \sin(xy)$$

80. Find the tangent plane equation to the function  $f(x, y) = z = 5 - x^2 - y^2$  at the point  $(1, 2)$ .

Solution:

$$f(1, 2) = 1 = z_0, f_x(1, 2) = -2, f_y(1, 2) = -4.$$

So

$$z - 1 = -2(x - 1) - 4(y - 2).$$

81. Consider  $z = \sqrt{11 + x^2 + y^2}$ . Use a linear approximation at  $(x, y) = (1, 2)$  to estimate the  $z$ -value when  $(x, y) = (2, 3)$ .

Solution: Note  $f(1, 2) = 4$ , and  $f_x = \frac{x}{\sqrt{11+x^2+y^2}}$ , and  $f_y = \frac{y}{\sqrt{11+x^2+y^2}}$ , therefore  $f_x(1, 2) = \frac{1}{4}$ , and  $f_y(1, 2) = \frac{1}{2}$ . The linear approximation

$$f(2, 3) = z \approx 4 + \frac{1}{4}(x - 1) + \frac{1}{2}(y - 2).$$

At  $(x, y) = (2, 3)$ , the approximate  $z$ -value is

$$z = 24 \approx 4 + \frac{1}{4} + \frac{1}{2} = 4 + \frac{3}{4} = 4.75$$

Note that the true value is  $\sqrt{24} = 4.8990\dots$

82. Consider the function  $f(x, y) = z = y \sin(x)$ , evaluated on the trajectory  $\sim r(t) = ht, t^2i$  from  $t = 0$  to  $\pi/2$ . Compute  $\frac{dz}{dt}$  at  $t = 1$ .

Solution:  $f_x = y \cos(x)$ ,  $f_y = \sin(x)$ ,  $x^0(t) = 1$ ,  $y^0(t) = 2t$ . So,

$$\frac{dz}{dt} = \pi/2 \cos(\pi/2)1 + \sin(\pi/2)2\pi/2 = \pi.$$

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83. (11.5) Consider

$$z = xy + e^{x^2}x(r, s, t) = rs + t y(r, s, t) = r^2s$$

Find the following expressions.

(a)  $\frac{\partial z}{\partial r}$

Solution:  $\frac{\partial z}{\partial r}$

$$y + 2xe^{x^2} s + x2rs$$

$$\frac{\partial z}{\partial r} =$$

$$\frac{\partial x}{\partial r}$$

$$\frac{\partial z}{\partial r} + \frac{\partial z}{\partial x}$$

$$\frac{\partial y}{\partial r}$$

$$\frac{\partial r}{\partial r} =$$

$$\frac{\partial x}{\partial r}$$

$$\frac{\partial y}{\partial r}$$

Plugging in  $x(r, s, t)$  and  $y(r, s, t)$ , we get:

$$r^2s + 2(rs + t)e^{(rs+t)^2} s + (rs + t)2rs.$$

$$\frac{\partial z}{\partial r} =$$

$$\frac{\partial z}{\partial r} =$$

(b)  $\frac{\partial^2 z}{\partial r^2}$

$\frac{\partial^2 z}{\partial r^2}$  (do this if covered in class)

Solution:

$$\frac{\partial^2 z}{\partial r^2} = \frac{\partial}{\partial r} (y + 2xe^{x^2} s + x2rs) = (2 + 4x^2)e^{x^2} s + 2rs s + s2rs + x2s. \frac{\partial z}{\partial r}$$

Then substitute  $x(s, t)$  and  $y(s, t)$  into the above.

84. Consider the function  $z = x^2 + 4y^2$ , evaluated on the trajectory  $\tilde{r}(t) = h^{-1}_2 \cos(t), \frac{1}{2} \sin(t) i$ .

- (a) Describe the trajectory graphically over  $t = 0$  to  $2\pi$ .
- (b) Compute  $\frac{dz}{dt}$ . The double angle formula may be useful:  $2 \sin(t) \cos(t) = \sin(2t)$ .
- (c) At what times  $t$  does  $\frac{dz}{dt} = 0$ . Which are max and min values?
- (d) Using the above, describe the motion of  $z(t)$  in 3D.

Solution:

- (a) It's a circular path with radius  $1/2$ .
- (b) Using the chain rule:  $dz/dt = -\cos(t) \sin(t) + 4 \sin(t) \cos(t) = 3 \sin(t) \cos(t) = \frac{3}{2} \sin(2t)$  (use double-angle formula)
- (c)  $\sin(2t) = 0$  when  $t = 0, \pi/2, \pi, 3\pi/2, 2\pi$ , that alternate from min, max, min, max, min, respectively.
- (d) At what times is  $\frac{dz}{dt}$  largest in absolute value?
- (e) It's like a up-down-up-down roller coaster.

85. (11.6) Consider the function  $f(x, y) = x^3 y - \sqrt{xy}$ .

- (a) Find the directional derivative at  $(2, 2)$  in direction of  $h-2, 1i$ .

Solution:

$$\tilde{\nabla} f(x, y) = h3x^2 y - y \quad \sqrt{xy}, x^3 - x$$

$$\tilde{\nabla} f(2, 2) = h23, 7i$$

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The unit vector in the direction of  $h-2, 1i$  is  $\frac{1}{5}h-2, 1i$ . So the directional derivative is  $h23, 7i \cdot \frac{1}{5}h-2, 1i = \frac{1}{5}(-46 + 7) = -\frac{39}{5}$

- (b) Find the directional derivative at  $(1, 4)$  in the direction of the vector given by the angle  $\theta = \frac{\pi}{3}$ .

Solution:

$$\tilde{\nabla} f(1, 4) = h10, 12i$$

So the directional derivative is

$$h10, 12i \cdot h\cos \frac{\pi}{3}, \sin \frac{\pi}{3}i = h10, 12i \cdot h\frac{1}{2}, \frac{\sqrt{3}}{2}i = 5 + \sqrt{3}4$$



86. (11.6) Find the direction, in which the tangent line of  $f(x, y) = xe^{x+y}$  at  $(1, -1)$  is the steepest.

Solution:

$$\nabla \tilde{f}(x, y) = h(x+1)e^{x+y}, xe^{x+y}; \nabla \tilde{f}(1, -1) = h2, 1i$$

So the tangent line is the steepest in the direction of  $h2, 1i$ .

87. (11.6) Find the tangent plane to the surface  $2x^2 - y^2 + 4z^2 = 8$  at  $(2, 2, 1)$ .

Solution: This surface is a level surface with

$$F(x, y, z) = 2x^2 - y^2 + 4z^2 = 8$$

So the normal vector of the tangent plane is

$$\nabla \tilde{F}(x, y, z) = h4x, -2y, 8zi \nabla \tilde{F}(2, 2, 1) = h8, -4, 8i = 4h2, -1, 2i$$

So the tangent plane is

$$2(x-2) - (y-2) + 2(z-1) = 0 \quad 2x - y + 2z = 4$$

88. (11.7) Consider the function  $f(x, y) = \frac{x-y}{x^2y^2+3}$ .

(a) Find all the critical points of  $f(x, y)$ .

Solution:

$$f_x(x, y) = x^2y^2 + 3 - (x-y)2xy^2$$

$$(x^2y^2 + 3)^2 f_y(x, y) = -(x^2y^2 + 3) - (x-y)x^22y$$

$$(x^2y^2 + 3)^2$$

Setting each equal to zero gives the equations

$$x^2y^2 + 3 - 2x^2y^2 + 2xy^3 = 0 \quad -x^2y^2 - 3 - 2x^3y + 2x^2y^2 = 0$$

$$3 - x^2y^2 + 2xy^3 = 0 \quad 3 - x^2y^2 + 2x^3y = 0$$

This gives  $xy^3 = x^3y$  and

$$0 = xy^3 - x^3y = xy(y^2 - x^2) = xy(y - x)(y + x)$$

Now clearly  $x = 0$  or  $y = 0$  does not solve the equations above. Now if  $y = x$ , both equations become

$$3 + x^4 = 0$$

These equations have no solution. If  $y = -x$ , both equations become

$$0 = 3 - 3x^4 = -3x^4 - 1 = -3x^2 + 1(x + 1)(x - 1)$$

So the critical points are  $(-1, 1)$  and  $(1, -1)$ .

(b) Decide whether the critical points are local minima, local maxima or saddle points.

Solution: To use the second derivative test, find the second derivatives

$$-2xy^2 + 2y^3 \quad x^2y^2 + 3 \quad -3 - x^2y^2 + 2xy^3 \quad 2xy^2$$

$$f_{xx}(x, y) =$$

$$(x^2y^2 + 3)^3$$

$$x^22y - 2x^3 \quad x^2y^2 + 3 \quad -3 + x^2y^2 - 2x^3y \quad x^22y$$

$$f_{yy}(x, y) =$$

$$(x^2y^2 + 3)^3$$

$$-x^22y + 6xy^2 \quad x^2y^2 + 3 \quad -3 - x^2y^2 + 2xy^3 \quad x^22y$$

$$f_{xy}(x, y) =$$

$$(x^2y^2 + 3)^3$$

$$f_{xx}(1, -1) = (-4)(4) - 0$$

$$4^3 = -4 f_{xx}(-1, 1) = (4)(4) - 0$$

$$4^3 = 14$$

$$4^3 = -4 f_{yy}(-1, 1) = (4)(4) - 0$$

$$f_{yy}(1, -1) = (-4)(4) - 0$$

$$4^3 = 14$$

$$f_{xy}(1, -1) = (8)(4) - 0$$

$$4^3 = 1 f_{xy}(-1, 1) = (-8)(4) - 0$$

$$\frac{3}{4} = -1$$

So it is

$$D(1, -1) = \frac{1}{16} - 1 < 0 \quad D(-1, 1) = \frac{1}{16} - 1 < 0$$

Thus both points are saddle points.

89. Find and characterized the critical points as local min, max, or saddles, of  $f(x, y) = 3xy - x^3 - y^3$  using the second derivative test.

Solution: First find the critical points

$$f_x(x, y) = 3y - 3x^2 \quad f_y(x, y) = 3x - 3y^2$$

Setting both zero gives

$$y = x^2 \quad x = y^2$$

Where these two parabolas intersect at  $(0, 0)$  and  $(1, 1)$  defining the critical points. The second derivative test

$$f_{xx} = -6x, \quad f_{yy} = -6y, \quad f_{xy} = f_{yx} = 3,$$

defining the  $D$  number

$$D(x, y) = 36xy - 9.$$

$D(0, 0) = -9 < 0$ , implying the origin critical point is a saddle point.

$D(1, 1) = 36 - 9 > 0$ , and  $f_{xx}(1, 1) = -6 < 0$  implying  $(1, 1)$  critical point is local maxima. The graph of the surface and the two critical points are shown in the figure: red dot is the saddle, with diagonal traces (black lines) showing one concave up curve and another concave down curve. The green dot is the local max, with both diagonal traces being concave down. Note, the function is unbounded as  $(x, y)$  gets increasingly far from the origin, but we are only concerned with local critical points.

90. (11.7) Find the absolute maximum and the absolute minimum of  $f(x, y) = xy^2 - x^2y + x$  on the closed rectangle with edges  $(0, 0)$ ,  $(0, 4)$ ,  $(3, 4)$  and  $(3, 0)$ .

Solution: First find the critical points

$$f_x(x, y) = y^2 - 2xy + 1 \quad f_y(x, y) = 2xy - x^2$$

Setting both zero gives

$$0 = y^2 - 2xy + 1 \quad 0 = 2xy - x^2 = x(2y - x)$$

From the second equation it is  $x = 0$  or  $x = 2y$ . If  $x = 0$ , then the second equation gives  $y^2 + 1 = 0$ . This has no solution. If  $x = 2y$ , the second equation is

$$0 = y^2 - 4y^2 + 1 = -3y^2 + 1 = -3\left(y - \frac{1}{\sqrt{3}}\right)\left(y + \frac{1}{\sqrt{3}}\right)$$

$$\frac{\sqrt[3]{2}}{3}, \frac{\sqrt[3]{1}}{3}$$

$$-\frac{\sqrt[3]{2}}{3}, -\frac{\sqrt[3]{1}}{3}$$

So the critical points are

and

. The function values at these points are

$$\frac{2\sqrt[3]{3}}{3}, \frac{1\sqrt[3]{3}}{3} = 2 - 4 + 6$$

$$-\frac{2\sqrt[3]{3}}{3}, -\frac{1\sqrt[3]{3}}{3} = -2 + 4 - 6$$

$$\frac{\sqrt[3]{3}}{3} = 4$$

$$\frac{\sqrt[3]{3}}{3} = -4$$

$$\frac{\sqrt[3]{3}}{3} f$$

$$\frac{\sqrt[3]{3}}{3}$$

$f$

Now check the boundary for extrema.

- $(0, 0) - (0, 4)$ :  $f(0, y) = 0$
- $(0, 4) - (3, 4)$ :  $f(x, 4) = 16x - 4x^2 + x = -4x^2 + 17x$ , which has critical point for  $8x = 17$ , that is at  $x = \frac{17}{8}$ . So the possible extrema are

$$f(0, 4) = 0 \quad f(3, 4) = 15 \quad f\left(\frac{17}{8}, 4\right) = \frac{17^2}{16}$$

- $(3, 4) - (3, 0)$ :  $f(3, y) = 3y^2 - 9y + 3$ , which has critical point for  $6y = 9$ , that is  $y = \frac{3}{2}$ . So the possible extrema are

$$f(3, 4) = 15 \quad f(3, 0) = 3 \quad f\left(3, \frac{3}{2}\right) = -\frac{15}{4}$$

- $(3, 0) - (0, 0)$ :  $f(x, 0) = x$ , so the possible extrema are

$$f(3, 0) = 3 \quad f(0, 0) = 0$$

So the absolute maximum is  $\frac{17^2}{16}$

and the absolute minimum  $-\frac{15}{4}$ .

91. (11.8) Find the maximum of  $z = 6x^2 + y$ , when  $y^2 + x^2 = 4$ .

Solution: Using Lagrange multipliers gives the equations

$$12x = \lambda 2x \quad 1 = \lambda 2y \quad x^2 + y^2 = 4$$

The first equation gives

$$0 = 6x - \lambda x = x(6 - \lambda)$$

So  $x = 0$  or  $\lambda = 6$ . If  $x = 0$ , then  $y = 2$  from the third equation. If  $\lambda = 6$ , then  $y = \frac{1}{12}$  from the second equation and the third equation gives  $x = \frac{\sqrt{575}}{12}$ .

The first gives the minimum and the second the maximum.

92. (11.8) Find the minimum of z-value of  $z = x^2 - y^2$ , subject to the constraint  $x + 2y = -1$ .

Solution: Using Lagrange multipliers

$$\nabla f = \langle 2x, -2y \rangle, \quad \nabla g = \langle 1, 2 \rangle$$

gives the equations

$$2x = \lambda, \quad -2y = 2\lambda$$

The first equation gives

$$x = \lambda/2.$$

and the second gives

$$-y = \lambda$$

so

$$x = -y/2$$

meaning the line  $y = -2x$  for any lambda-value solves the Lagrange equation. The line  $y = -2x$  satisfies/intersects the constraint equation  $x + 2y = -1$  when  $x - 4x = -1$ . So,  $x = 1/3$  and  $y = -2/3$  is the critical point and  $\lambda = 2/3$ . This point is a minimum  $z = 1/9 - 4/9 = -1/3$ . A figure of the saddle surface  $z = x^2 - y^2$  overlaid by constraint equation line  $x + 2y = -1$  (black line) shows that at  $(1/3, -2/3)$  the minimum point  $z = -1/3$  is attained (green dot). For reference, the saddle location at the origin (red dot) is also shown.

93. (12.1) Given the function  $f(x, y) = x^2 + y$  over the rectangle  $R = [1, 3] \times [2, 5]$ . Divide the rectangle into smaller rectangle, by dividing it into eight pieces in x-direction and six pieces in y-direction. Set up the Riemann sum using the midpoint rule to estimate the volume of the function over  $R$ . Do not evaluate the sum.

Solution:

$$8 = \sum_{k=1}^4 \Delta x_k = 1 + \frac{1}{4}$$

$$N = 8, \Delta x = 3 - 1$$

$$6 = \sum_{l=1}^2 \Delta y_l = 2 + \frac{1}{2}$$

$$M = 6, \Delta y = 5 - 2$$

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The sample points using the midpoint rule are

$$x_{kl}^* = x_k + x_{k-1} \quad \text{and} \quad y_{kl}^* = y_l + y_{l-1}$$

$$2 = 1 + \frac{k-1}{4} \quad \text{and} \quad 2 = 2 + \frac{l-1}{4}$$

$$2 = 2 + \frac{l-1}{4}$$

So the Riemann sum is

$$1 + \frac{k-1}{4} \cdot \frac{2}{8} + 2 + \frac{l-1}{4} \cdot \frac{1}{8}$$

$$V \approx X^8$$

$$X^6$$

$$f(x_{kl}^*, y_{kl}^*) \Delta x \Delta y = X^8$$

$$X^6$$

$$k=1$$

$$l=1$$

$$k=1$$

$$l=1$$

94. (12.2) Find the volume of the solid that lies above the square  $R = [0, 2] \times [0, 4]$  and below the surface defined by  $z = ye^{xy}$ .

Solution: To integrate this function it is better to first integrate with respect to  $x$ . That gives  $Z$

$$V = \int_R ye^{xy} dA = \int_0^4 \int_0^2 ye^{xy} dx dy$$

$$= \int_0^4 [e^{xy}]_{x=0}^2 dy = \int_0^4 (e^{2y} - 1) dy = \left[ \frac{1}{2}e^{2y} - y \right]_0^4$$



$$= \frac{1}{2}e^8 - 8 - \frac{1}{2}$$

95. (12.3) Evaluate  $\iint_R xy \, dA$ , where  $D$  is the region bounded by  $x = y^2$  and  $y = -x + 2$ .

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Solution: To describe the region, first sketch it

To find the intersection points, set the equations, after solving both for  $x$ , equal

$$y^2 = x = y + 2 \Rightarrow 0 = y^2 + y - 2 = (y - 1)(y + 2).$$

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Then the intersection points are  $(1, 1)$  and  $(4, 2)$ , so  $=^2$

To find the intersection points, set the equations, after solving both for  $x$ ,

$$\text{equal } y^2 = x = -y + 2 \implies 0 = y^2 + y - 2 = (y - 1)(y + 2).$$

Then the intersection points are  $(1, 1)$  and  $(4, -2)$ , so

$$D = \{(x, y) \mid -2 \leq y \leq 1, y^2 \leq x \leq -y + 2\}.$$

Then the integral gives

$$\begin{aligned} \iint_D xy \, dA &= \int_{-2}^1 \int_{y^2}^{-y+2} xy \, dx \, dy \\ &= \int_{-2}^1 \left[ \frac{1}{2} x^2 y \right]_{x=y^2}^{x=-y+2} dy \\ &= \int_{-2}^1 \left( \frac{1}{2} (-y+2)^2 y - \frac{1}{2} y^4 \right) dy \\ &= \int_{-2}^1 \left( \frac{1}{2} (y^2 - 4y + 4) y - \frac{1}{2} y^4 \right) dy \\ &= \int_{-2}^1 \left( \frac{1}{2} y^3 - 2y^2 + 2y - \frac{1}{2} y^4 \right) dy \\ &= \left[ \frac{1}{8} y^4 - \frac{2}{3} y^3 + y^2 - \frac{1}{10} y^5 \right]_{-2}^1 \\ &= \left( \frac{1}{8} - \frac{2}{3} + 1 - \frac{1}{10} \right) - \left( \frac{16}{8} - \frac{16}{3} + 4 - \frac{32}{10} \right) \\ &= \frac{1}{8} - \frac{2}{3} + 1 - \frac{1}{10} - \left( 2 - \frac{16}{3} + 4 - \frac{32}{10} \right) \\ &= \frac{1}{8} - \frac{2}{3} + 1 - \frac{1}{10} - 2 + \frac{16}{3} - 4 + \frac{32}{10} \\ &= \frac{1}{8} - \frac{2}{3} + \frac{16}{3} - 3 + \frac{32}{10} - \frac{1}{10} \\ &= \frac{1}{8} + \frac{14}{3} - 3 + \frac{31}{10} \\ &= \frac{1}{8} + \frac{140}{30} - \frac{60}{30} + \frac{93}{30} \\ &= \frac{1}{8} + \frac{173}{30} \\ &= \frac{1}{8} + \frac{173}{30} = \frac{15}{240} + \frac{865}{240} = \frac{880}{240} = \frac{11}{3} \end{aligned}$$

96. (12.3) Integrate the function  $f(x, y) = xy$  over the set  $D = \{(x, y) \mid 0 \leq x \leq 1, y \geq x, y \leq 4\}$ . Choose the

- |   |   |
|---|---|
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| n | r |
| t | d |
| e | e |
| g | r |
| r | s |
| a | o |
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| i | h |
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R  
D

$f(x, y) dA$  in only a single iterated integral.

Solution:

$$\iint_D f(x, y) dA = \int_0^1 \int_x^4 xy \, dy \, dx = \int_0^1 \left[ \frac{xy^2}{2} \right]_x^4 dx = \int_0^1 \left( \frac{16x}{2} - \frac{x^3}{2} \right) dx = \int_0^1 (8x - \frac{1}{2}x^3) dx = \left[ 4x^2 - \frac{1}{8}x^4 \right]_0^1 = 4 - \frac{1}{8} = \frac{31}{8}$$

97. Integrate the function  $f(x, y) = xy$  over the set  $D = \{(x, y) | 0 \leq x \leq 1, y \geq x, y \leq 4\}$ . Choose the integration order so that can you compute the integral  $\iint_D f \, dA$  in only a single iterated integral. Do not compute the integral.

Solution:

$$\int_0^1 \int_x^4 xy \, dy \, dx$$

98. Suppose a lamina object with boundaries  $y = x$  and  $y = \sqrt{x}$  has a density function  $\rho(x, y) = y(1-x^2/2)$ . Find the mass of the object.

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Solution:

$$\int_0^1 \int_x^{\sqrt{x}} y(1-x^2/2) dy dx = 17/240$$

99. (12.4) Write the set  $D = \{(x, y) | x \geq 0, y \leq 0, x^2 + y^2 \leq 4\}$  in polar coordinates.

Solution: This is a quarter of a circle in the fourth quadrant, so

$$D = \{(r, \theta) | r \leq 2, -\pi/2 \leq \theta \leq 0\}$$

or equivalently

$$D = \{(r, \theta) | r \leq 2, 3\pi/2 \leq \theta \leq 2\pi\}$$

100. (12.4) Using polar coordinates evaluate  $\iint_D f(x, y) dA$ , where  $D$  is a disk of radius 2 and  $f(r, \theta) = r^2 \cos(2\theta)$ .

Solution: The region  $D$  can be described as  $D = \{(r, \theta) | r \leq 2\}$ .

$$\iint_D f(x, y) dA = \int_0^{2\pi} \int_0^2 r^2 \cos(2\theta) \cdot r dr d\theta = \int_0^{2\pi} \cos(2\theta) d\theta = \frac{1}{2} \sin(2\theta) \Big|_0^{2\pi} = 0$$

$$= 2^4 \int_0^{2\pi} \int_0^4 r \, dr \, d\theta = 0$$

101. (12.4) Find the area of the flower-shaped region

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 $D = \{(r, \theta) \mid r \leq \sin(8\theta) + 4\}$ .

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$$\int_0^{2\pi} \int_0^{\sin(8\theta)+4} r \, dr \, d\theta$$

Solution: Use polar coordinates:  $dA = r \, dr \, d\theta$

$$A = \int_0^{2\pi} \int_0^{\sin(8\theta)+4} r \, dr \, d\theta = \int_0^{2\pi} \left[ \frac{1}{2} r^2 \right]_0^{\sin(8\theta)+4} d\theta$$

$$= \int_0^{2\pi} \frac{1}{2} (16 + 8\sin(8\theta) + (\sin(8\theta))^2) d\theta$$

since  $\int_0^{2\pi} \cos(\theta) \, d\theta = 0$  and  $\int_0^{2\pi} \sin(\theta) \, d\theta = 0$ .

8. (12.5) Suppose a laminar object with boundaries  $y = x$  and  $y = \frac{1}{2}x$  has a density function

$\rho(x, y) = y$   
 Find the mass of the object.

Solution: Use polar coordinates:  $dA = r dr d\theta$

$$\begin{aligned}
 A &= \int_0^{2\pi} \int_0^{\sin(8\theta)+4} r dr d\theta \\
 &= \int_0^{2\pi} \left[ \frac{1}{2} r^2 \right]_0^{\sin(8\theta)+4} d\theta \\
 &= \int_0^{2\pi} \frac{1}{2} (16 + 8 \sin(8\theta) + (\sin(8\theta))^2) d\theta \\
 &= \int_0^{2\pi} (8 + 4 \sin(8\theta) + \frac{1}{4} - \frac{1}{4} \cos(16\theta)) d\theta = \frac{33}{4} \pi
 \end{aligned}$$

since  $\int_0^{2\pi} \cos(\theta) d\theta = 0$  and  $\int_0^{2\pi} \sin(\theta) d\theta = 0$ .

102. (12.5) Suppose a lamina with boundaries  $y = x$  and  $y = \sqrt{x}$  has a density function  $\rho(x, y) =$  mass of the object.

$y$

$1 - \frac{x_2}{2}$ . Find the

Solution: The intersections of the boundaries are at (0, 0) and (1, 1). So the region on which the object lies can be described as

$$D = (x, y) \quad 0 \leq x \leq 1, \quad x \leq y \leq \sqrt{1-x^2}$$

$$\sqrt{1-x^2}$$

$$\int_0^1 \int_x^{\sqrt{1-x^2}} dy \, dx = Z_1$$

1

Z<sub>1</sub>

Z<sub>1</sub>

$$2y^2$$

m =

y

dx

0

x

0

y=x

$$\int_0^1 \left( \frac{1}{2}x^3 - \frac{1}{2}x^2 - \frac{1}{2}x^4 \right) dx = \left[ \frac{1}{8}x^4 - \frac{1}{6}x^3 - \frac{1}{10}x^5 \right]_0^1 = \frac{1}{8} - \frac{1}{6} - \frac{1}{10} = \frac{5}{240} - \frac{4}{240} - \frac{24}{240} = -\frac{23}{240}$$

$$= \frac{1}{2} Z_1$$

$$= \frac{1}{2} \left( \frac{1}{2} - \frac{1}{8} - \frac{1}{3} + \frac{1}{10} \right) = 17$$

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103. Some objects can have the center of mass outside itself, e.g., a boomerang. Let's consider an idealized boomerang with constant density and the shape shown in the figure. Find its center of mass.

104. (12.5) Let  $p(x, y)$  be the probability density function defined for the random variables  $X$  and  $Y$  in the unit square  $D = [0, 1] \times [0, 1]$ , given by

$$p(x, y) = 4y(1 - x).$$

(a) Check that this function is a probability density function.

Solution:

$$\int_0^1 \int_0^x 4y(1-x) dy dx = 4 \int_0^1 \left[ \frac{1}{2} y^2 (1-x) \right]_0^x dx = 2 \int_0^1 (1-x)^2 dx = 2 \left[ -\frac{1}{3} (1-x)^3 \right]_0^1 = 2 \left( 0 - \left(-\frac{1}{3}\right) \right) = \frac{2}{3} \neq 1$$

So this function is not a probability density function.

(b) Set up the iterated integral for the probability that  $(x, y) \in A$ , where  $A$  contains all points  $(x, y)$  in  $D$  for which  $x$  is greater than  $y$ . Do not compute the integral.

Solution: The region  $A$  can be described as

$$A = \{(x, y) | 0 \leq x \leq 1, 0 \leq y \leq x\}.$$

Then the probability is

$$P((x, y) \in A) = \int_0^1 \int_0^x 4y(1-x) dy dx.$$

105. (12.5) Suppose a factory must perform two operations on items they are producing. The time (in hours) it takes for operation 1 is  $X$  and for operation 2 it is  $Y$ . The joint probability distribution is

$$p(x, y) =$$



$$4xye^{-x^2-y^2} \geq 0, y \geq$$

$$0 \text{ otherwise}$$

What is the probability that operation 1 takes more than 2 hours, AND operation 2 takes more than 2 hours?

Solution: The region that describes this occurrence is

$$D = \{(x, y) | x \geq 2, y \geq 2\}$$

Then the region  $D$  cannot be integrated in only one iterated integral. Therefore it is easier to integrate the region outside of  $D$ , call it  $R = D = \{(x, y) | x \leq 2, y \leq 2\}$  and subtract the result from

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1, to get the probability

$$\begin{aligned}
 P((x, y) \in D) &= 1 - P((x, y) \in R) \\
 &= 1 - \int_0^2 \int_0^2 4xye^{-x^2-y^2} dy dx \\
 &= 1 - \int_0^2 \left[ -2xe^{-x^2-y^2} \right]_0^2 dx \\
 &= 1 - \int_0^2 (-2xe^{-x^2-4} + 2xe^{-x^2}) dx \\
 &= 1 - \left[ -e^{-x^2-4} + e^{-x^2} \right]_0^2 \\
 &= 1 - \left( -e^{-4} + e^{-4} + e^{-4} - e^{-4} \right) \\
 &= 1 - 0 \\
 &= 1
 \end{aligned}$$

The integral  $\int_0^2 2ye^{-y^2} dy = 1 - e^{-4}$ , so

$$P((x, y) \in D) = 1 - (1 - e^{-4})^2$$